# NON-INTERSECTING PATHS, RANDOM TILINGS AND RANDOM MATRICES

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ABSTRACT. We investigate certain measures induced by families of non-intersecting paths in domino tilings of the Aztec diamond, rhombus tilings of an abc-hexagon, a dimer model on a cylindrical brick lattice and a growth model. The measures obtained, e.g. the Krawtchouk and Hahn ensembles, have the same structure as the eigenvalue measures in random matrix theory like GUE, which can in fact can be obtained from non-intersecting Brownian motions. The derivations of the measures are based on the Karlin-McGregor or Lindström-Gessel-Viennot method. We use the measures to show some asymptotic results for the models.

## 1. INTRODUCTION

We begin by summarizing some facts about the Gaussian Unitary Ensemble of random hermitian matrices. We will see analogues of these in the random tiling problems discussed below. The *Gaussian Unitary Ensemble* (GUE) is the probability measure

$$\frac{1}{\mathcal{Z}_N} e^{-\operatorname{tr} M^2} dM \tag{1.1}$$

on the space of all $N \times N$ hermitian matrices, which is isomorphic to $\mathbb{R}^{N^2}$, and $dM$ is the Lebesgue measure on this space. The induced measure on the $N$ real eigenvalues $\lambda_1, \dots, \lambda_N$ of $M$ is given by, [52],

$$\phi_{N,\mathrm{GUE}}(\lambda) d^N\lambda = \frac{1}{Z_N} \Delta_N(\lambda)^2 \prod_{j=1}^{N} e^{-\lambda_j^2} d^N\lambda \tag{1.2}$$

where

$$\Delta_N(\lambda) = \det(\lambda_j^{N-k})_{j,k=1}^N = \prod_{1\le j<k\le N} (x_j - x_k). \tag{1.3}$$

The probability of finding $m$ eigenvalues in infinitesimal intervals $d\lambda_1, \dots, d\lambda_m$ around $\lambda_1, \dots, \lambda_m$ is given by $R_{m,N}(\lambda_1, \dots, \lambda_m) d\lambda_1, \dots, d\lambda_m$, where $R_{m,N}$ is the *m-point correlation function*

$$R_{m,N}(\lambda_1, \dots, \lambda_m) = \frac{N!}{(N-m)!} \int_{\mathbb{R}^{N-m}} \phi_{N,\mathrm{GUE}}(\lambda_1, \dots, \lambda_N) d\lambda_{m+1}, \dots, d\lambda_N. \tag{1.4}$$

The correlation functions are given by determinants,

$$R_{m,N}(\lambda_1, \dots, \lambda_m) = \det(K_N(\lambda_i, \lambda_j))_{i,j=1}^m, \tag{1.5}$$

where

$$K_N(x,y) = \frac{\kappa_{N-1}}{\kappa_N} \frac{h_N(x)h_{N-1}(y) - h_{N-1}(x)h_N(y)}{x-y} \left(e^{-x^2-y^2}\right)^{1/2}, \tag{1.6}$$

and $h_n(x) = \kappa_n x^n + \dots$ are the orthonormal Hermite polynomials.

We can think of the eigenvalues as points on $\mathbb{R}$. Using the formulas above and asymptotics for Hermite polynomials we can obtain a limiting random point process with correlation functions

$$R_m(x_1,\dots,x_m) = \det\left(\frac{\sin\pi(x_i-x_j)}{\pi(x_i-x_j)}\right)_{i,j=1}^m. \tag{1.7}$$

Hence we obtain a determinantal random point process given by the sine kernel, [62]. To get this limit we rescale so that the mean distance between the eigenvalues (points) equals 1 and then use,

$$\lim_{N\to\infty} \frac{1}{\sqrt{2N}\rho(u)} K_N\left(\sqrt{\frac{N}{2}}u + \frac{\xi}{\sqrt{2N}\rho(u)}, \sqrt{\frac{N}{2}}u + \frac{\eta}{\sqrt{2N}\rho(u)}\right) = \frac{\sin\pi(\xi-\eta)}{\pi(\xi-\eta)}, \tag{1.8}$$

where $\rho(u) = \frac{1}{2\pi}\sqrt{(4-u^2)_+}$ is the *Wigner semicircle law* which describes the asymptotic density of the eigenvalues.

Let us recall two limit theorems for the fluctuations of the eigenvalues. Denote by $\#(u,v)$ the number of eigenvalues in the interval $[u\sqrt{N/2}, v\sqrt{N/2}]$, $u < v$, $|u|,|v| < 2$. Then,

$$\frac{\#(u,v) - \mathbb{E}_N[\#(u,v)]}{\sqrt{\frac{2}{\pi^2}\log N}} \tag{1.9}$$

converges in distribution to the standard normal, [11], [61], [62]. If $\lambda_{\max} = \max_{1\le j\le N}\lambda_j$ is the largest eigenvalue, then

$$\mathbb{P}_N\left[\frac{\lambda_{\max} - \sqrt{2N}}{\sqrt{2}N^{-1/6}} \le \xi\right] \to F(\xi) \tag{1.10}$$

as $N\to\infty$, $\xi\in\mathbb{R}$, where

$$F(\xi) = \det(I-A)_{L^2(\xi,\infty)}. \tag{1.11}$$

Here $A$ is the operator on $L^2(\xi,\infty)$ with kernel (the *Airy kernel*)

$$A(x,y) = \frac{\mathrm{Ai}\,(x)\mathrm{Ai}\,'(y) - \mathrm{Ai}\,'(x)\mathrm{Ai}\,(y)}{x-y}, \tag{1.12}$$

[66]. The distribution function (1.11) is called the *Tracy-Widom distribution.* It follows from (1.5) and the Fredholm expansion that

$$\begin{aligned}\mathbb{E}_N\left[\prod_{j=1}^N (1+g(\lambda_j))\right] &= \sum_{n=0}^N \frac{1}{n!}\int_{\mathbb{R}^n} \det(K_N(x_i,x_j)g(x_j))_{i,j=1}^n d^n x \\ &= \det(I+K_N g)_{L^2(\mathbb{R})}.\end{aligned} \tag{1.13}$$

If we take $g(x) = -\chi_{(t,\infty)}(x)$ in (1.13) and use the asymptotics of the Hermite polynomials close to the largest zero, we can prove (1.10).

The GUE eigenvalue measure, (1.2), was obtained above from the measure (1.1) on random hermitian matrices. We will now show how we can obtain (1.2) in a completely different way using non-intersecting Brownian motions. This type of problem has been studied under the name of vicious walkers or domain walls in the statistical physics litterature, see [16], [19], [21], [20] and references in these papers.

Consider $N$ 1-dimensional Brownian motions starting at the points $0,\dots,N-1$ at time 0 and ending at the same points $0,\dots,N-1$ at time $2T$. Let $p_{N,T}(x_1,\dots,x_N)$ denote the probability density that at time $T$ the particles are at the positions $x_1<\dots<x_N$ conditioned on the event that the paths have not intersected in the whole time interval $[0,2T]$. If $p_t(x,y)=(2\pi t)^{-1/2}\exp(-(x-y)^2/2t)$ is the transition kernel for Brownian motion, then, by a theorem of Karlin and McGregor, [43], see also [40],

$$
\begin{aligned}
p_{N,T}(x_1,\dots,x_N) &= \frac{1}{Z_N}\det(p_T(j-1,x_k))_{j,k=1}^N \det(p_T(x_j,k-1))_{j,k=1}^N \\
&= \frac{1}{Z_N}\left(\det(p_T(j-1,x_k))_{j,k=1}^N\right)^2,
\end{aligned}
\tag{1.14}
$$

where

$$
Z_N=\frac{1}{N!}\int_{\mathbb{R}^N}\left(\det(p_T(j-1,x_k))_{j,k=1}^N\right)^2 d^Nx. \tag{1.15}
$$

Note that, because of symmetry, we can consider (1.14) as a probability measure on $\mathbb{R}^N$ and remove the $N!$ in (1.15). It follows from [67] that the probability measure on $\mathbb{R}^N$ with density (1.14) has determinantal correlation functions analogous to (1.5) but with a different kernel, see [40]. We can now obtain GUE as follows. Let $T\to\infty$ and rescale the $x_j$:s by $\sqrt{T}$ so that they do not move away to infinity. Then,

$$
\lim_{T\to\infty} p_{N,T}(x_1\sqrt{T},\dots,x_N\sqrt{T})=\phi_{N,\mathrm{GUE}}(x). \tag{1.16}
$$

To show this note that,

$$
\det(p_T(j-1,x_k))_{j,k=1}^N=\frac{1}{(2\pi t)^{-N/2}}\prod_{j=1}^N e^{((j-1)^2+x_j^2)/2T}\det(e^{(j-1)x_k/2T})_{j,k=1}^N \tag{1.17}
$$

and use the formula for a Vandermonde determinant. The choice of initial and final positions made above is not necessary for the result but simplifies the computations. Thus, we need not look upon (1.2) as something which necessarily comes from random matrices. For another relation between random matrices and Brownian motion see [3], [29].

The present paper can be seen as a continuation of the paper [38] where several examples of analogues of GUE on a discrete space was given. In these ensembles we have analogues of the results for GUE discussed above but we obtain the so called discrete sine kernel, in the limit instead of the ordinary sine kernel. The common theme of the present paper is non-intersecting paths, which are discrete analogues of the non-intersecting Brownian motions just mentioned, and thus it is reasonable to expect that we will have features analogous to those of GUE. In section 2 we will consider random tilings of the Aztec diamond of size $n$, [15], which can be described using certain non-intersecting paths. By using so called zig-zag paths in the tiling, we obtain the Krawtchouk ensemble, [38], which can be used to analyze several properties of the random tiling. By a result in [36] the shape of the so called temperate region in a random tiling is closely related to the corner growth model in [37], which is a generalization of the longest increasing subsequence problem for random permutations. This gives a new approach to the asymptotic fluctuation results in [2] and [37] involving the Tracy-Widom distribution. We will

also study the fluctuations of the domino height function which describes the diling. It has Gaussian fluctuations with variance of order $\log N$, a fact that is related to the Gaussian fluctuations in the number of eigenvalues in an interval in a GUE matrix. This type of result has been conjectured in [59]. We will also discuss the relation between the equilibrium measure for the Krawtchouk ensemble and the arctic ellipse. The corner growth model can be generalized further and this leads to the so called Schur measure introduced on [55]. The Schur measure can also be analyzed using non-intersecting paths as will be demonstrated in section 3. We will also, in section 4, consider rhombus tilings of a hexagon, [12], which are related to boxed plane partitions. These tilings can also be described by certain non-intersecting random walk paths and the intersection of these paths with a fixed line leads to the so called Hahn ensemble. In this problem we will not compute the detailed asymptotics, but we will discuss the equilibrium measure for the Hahn ensemble and its relation to the arctic ellipse phenomenon. Finally, in section 5, we will analyze certain aspects of a dimer model on a brick (hexagonal) lattice on a cylinder. Here we also have non-intersecting paths but the number of paths is not fixed like in the other examples. The methods used are very close to the arguments used to compute the correlation functions in [67]

There are many papers in the statistical physics litterature related to the present paper, e.g. [7], [16], [19], [21], [20], [30], [44], [49], [69] and [70] Connections between random permutations and the so called random turns model, which gives certain non-intersecting paths has beeen discussed in [1], [22], and [23]. Other relevant papers are [18], [24], [25], [28], [34] and [58].

## 2. The Aztec diamond

2.1. **Basic definitions.** The *Aztec diamond*, $A_n$, of order $n$ is the union of all lattice squares $[m, m+1] \times [l, l+1]$, $m, l \in \mathbb{Z}$, that lie inside the region $\{(x, y)\,;\, |x| + |y| \leq n+1\}$. A *domino* is a closed $1 \times 2$ or $2 \times 1$ rectangle in $\mathbb{R}^2$ with corners in $\mathbb{Z}^2$, and a *tiling* of a region $R \subseteq \mathbb{R}^2$ by dominoes is a set of dominoes whose interiors are disjoint and whose union is $R$. Let $\mathcal{T}(A_n)$ denote the set of all domino tilings of the Aztec diamond.

We can equivalently think of a tiling as a dimer configuration. Consider the graph $G$ with vertices at $(\frac{1}{2}, \frac{1}{2}) + \mathbb{Z}^2$ and edges between nearest neighbour vertices. A *dimer* is simply an edge in $G$, and if the edge goes between the verices $v_1$ and $v_2$ we say that the dimer *covers* $v_1$ and $v_2$. Let $G_n$ be the subgraph of $G$ where all vertices lie in $A_n$. A *dimer configuration* in $G_n$ is a set of dimers in $G_n$ such that all vertices are covered by exactly one dimer. This is clearly equivalent to a tiling of $A_n$ via the identification: a dimer between $v_1$ and $v_2$ corresponds to a domino covering the two lattice squares with centers $v_1$ and $v_2$.

Colour the Aztec diamond in a checkerboard fashion so that the leftmost square in each row in the top half is white. A horizontal domino is *north-going* (N) if its leftmost square is white, otherwise it is *south-going* (S). Similarly, a vertical domino is *west-going* (W) if its upper square is white, otherwise it is *east-going* (E). Two dominoes are *adjacent* if they share an edge, and a domino is adjacent to the boundary if it shares an edge with the bundary of the Aztec diamond. The *north polar region* is defined to be the union of those north-going dominoes that are connected to the boundary by a sequence of adjacent north-going dominoes. The south, west and east polar regions are defined analogously. In this way a domino

tiling partitions the Aztec diamond into four polar regions, where we have a regular brick wall pattern, and a fifth central region, the *temperate zone*, where the tiling pattern is irregular.

We will now define a one-to-one mapping from $\mathcal{T}(A_n)$ to families of $n$ non-intersecting lattice paths. Consider an S-domino which we place with corners at $(0,0),(2,0),(2,1),(0,1)$. Draw a straight line from $(0,1/2)$ to $(2,1/2)$. In this way we get a piece of a path, and we do this for all S-dominoes. Similarly we can put a W-domino so that it has corners at $(0,0),(1,0),(1,2),(0,2)$, and then draw a straight line segment from $(0,1/2)$ to $(1,3/2)$. Finally, on an E-domino placed at the same position we draw a straight line from $(0,3/2)$ to $(1,1.2)$. Do this for all W- and E-dominoes. We do not draw any line on an N-domino. Given a domino tiling of $A_n$ we draw lines on the dominoes as just described. We claim that this gives $n$ non-intersecting paths starting at $A_j=(n+1-j,\frac{1}{2}-j)$ and ending at $E_j=(-n-1+j,\frac{1}{2}-j)$, $1\le j\le n$. We call these paths *DR-paths of type I* (after D. Randall, [64], p.277). To prove the claim we argue as follows. Consider the black lattice square to the right of $E_j$. It has to be covered by a W- or am S-domino. In both cases a path will end at $E_j$ From the checkerboard colouring we see that we nust obtain connected paths. Similarly, if we consider the white lattice square to the left of $A_j$ it can only be covered by an E- or an S-domino. Hence a path must start at the point $A_j$. Clearly, by construction, the paths are non-intersecting and the claim is proved.

A convenient coordinate system for describing the paths is what we call *coordinate system I* (CS-I). As origin we take $(n+1,1/2)$ and as basis vectors $\mathbf{e}_I=(-1,-1)$, $\mathbf{f}_I=(-1,1)$. Let $\mathcal{L}_I$ be the integer lattice in CS-I. The type I DR-paths are walks in $\mathcal{L}_I$. They take steps $(1,0)$, $(0,1)$ or $(1,1)$ and they have starting points $(k,0)$ and endpoints $(n+1,k)$, $1\le k\le n$. Thus we obtain a map from domino tilings of $A_n$ to families of $n$ nonintersecting type I DR-paths in $\mathcal{L}_I$ with the specified initial and final positions. This map is a bijection. To see this, fill in with dominoes along the paths using the marked tiles. This is possible since the paths do not intersect. If we have a white lattice square that is covered by a domino, then the black square to the right is also empty, since otherwise the paths would not be connected. Similarly, if a black square is not covered by a domino, the white square to the left is not covered either. Hence, the squares that are not already covered can be covered by N-dominoes. Clearly this gives an inverse.

We can also define *type II DR-paths* which are complementary to the type I paths. In this case the S-dominoes are unmarked, whereas the N-dominoes have a horizontal segment in the middle. Furthermore we interchange the marking on the W- and E-dominoes. In this way we obtain paths from $A_j=(-n-1+j,j-1/2)$ to $E_j=(n+1-j,j-1/2)$, $1\le j\le n$. In coordinate system II (CS-II) which has origin $(-n-1,-1/2)$ and basis vectors $\mathbf{e}_{II}=(1,1),\mathbf{f}_{II}=(1,-1)$, we obtain the same type of lattice paths as before.

Call the top type I DR-path the *level-1 path*. It is clear from the definitions above that the north polar zone is exactly the part of the Aztec diamond above the level-1 path, i.e. all dominoes, which have to be N-domonioes, that lie above this path.

Let $\tau\in\mathcal{T}(A_n)$ be a tiling of the Aztec diamond and let $v(\tau)$ denote the number of vertical dominoes in $\tau$. We define a probability measure on $\mathcal{T}(A_n)$ by letting the

horizontal dominoes have weight 1 and the vertical dominoes weight $w$. Thus,

$$\mathbb{P}[\tau] = \frac{w^{v(\tau)}}{\sum_{\tau \in \mathcal{T}(A_n)} w^{v(\tau)}}. \tag{2.1}$$

If we take $w = 1$ we obtain the uniform distribution on $\mathcal{T}(A_n)$. This can alternatively be viewed as a probability measure on the DR-paths, where we put the weight 1 on the steps $(1, 1)$ (in CS-I or CS-II), which correspond to horizontal dominoes, and the weight $w$ on the steps (1,0) or (0,1), which correspond to vertical dominoes. The weight of $n$ given non-intersecting DR-paths is the product of the weights on all steps and equals $w^{v(\tau)}$, if $\tau$ is the tiling defined by the paths. The weight of a set of non-intersecting DR-paths is the sum of the weights of all the elements in the set.

Next, we will define another type of paths, the so called zig-zag paths, [15], in the Aztec diamond. Consider the sequence of white squares with opposite corners $Q^r_k = (-r + k, n + 1 - k - r)$, $k = 0, \dots, n + 1$, where $r$, $1 \le r \le n$, is fixed. A *zig-zag path* $Z_r$ in $A_n$ is a path of edges going around these white squares. When going from $Q^r_k$ to $Q^r_{k+1}$ we can go either first one step east and then one step south (an ES-step), or first one step south and then one step east (an SE-step). A domino tiling $\tau \in \mathcal{T}(A_n)$ defines a unique zig-zag path $Z_r(\tau)$ from $Q^r_0$ to $Q^r_{n+1}$ if we require that the zig-zag path does not intersect the dominoes. There will be exactly $r$ ES-steps, and hence $n + 1 - r$ SE-steps along the zig-zag path. This can be proved using the domino height function defined below.

We associate the point $(r, n-k)$ in CS-I with the step $Q^r_k Q^r_{k+1}$. Suppose that we have ES-steps at the points $(r, h_j)$, $1 \le j \le r$, in CS-I, i.e. $Q^r_{n-h_j} Q^r_{n-h_j+1}$ are ES-steps. Then the zig-zag path is mapped one-to-one to $(h_1, \dots, h_r)$, where $0 \le h_1 < \cdots < h_r \le n$. This specifies the *zig-zag (particle) configuration* $(h_1, \dots, h_r)$; we write $p(Z_r) = (h_1, \dots, h_r)$. We can also associate the step $Q^r_k Q^r_{k+1}$ with the point $(n+1-r, k)$ in CS-II, and we will then have SE-steps at the points $(n+1-r, n-k_j)$, $1 \le j \le n + 1 - r$, where $k_1 < \cdots < k_{n+1-r}$. We call $(k_1, \dots, k_{n+1-r})$ the *zig zag (hole) configuration*, and write $h(Z_r) = (k_1, \dots, k_{n+1-r})$. The next lemma gives the relation between the DR-paths and the zig-zag paths.

**Lemma 2.1.** *The points $(r, h_j)$ in CS-I are the last positions on $x_I = r$ of the type I DR-paths starting at $(k, 0)$, $1 \le k \le r$ in CS-I. Similarly, $(n + 1 - r, n - k_j)$ are the last positions on $x_{II} = n + 1 - r$ in CS-II of the type II DR-paths starting at $(k, 0)$, $1 \le k \le n + 1 - r$. Also,*

$$\{h_1, \dots, h_r\} \cup \{k_1, \dots, k_{n+1-r}\} = \{0, \dots, n\}. \tag{2.2}$$

*Proof.* If we have an ES-step around a white square, then this square is covered by an S- or a W-domino. In both cases it follows, from the definition of the type I DR-paths, which are walks in the integer lattice $\mathcal{L}_I$ in CS-I, that the DR-path must intersect the S-step in the ES-step, and after that go to a point with a larger $x_I$-coordinate. Similarly, if we have an SE-step around a white square, then this square is covered by an N- or an E-domino, in which case a type I DR-path does not intersect neither the S- nor the E-step. The proof of the second statement in the lemma is analogous, and (2.2) follows from the definition of the particle and hole configurations, and the definition of CS-I and CS-II. □

If $p[Z_r(\tau)] = (h^r_1, \dots, h^r_r)$, $1 \le r \le n$, then the position of the rightmost particle, $h^r_r$, describes the level-1 type I DR-path. As noted above, the region above this

DR-path is the north polar zone, and hence we can investigate the shape of the north polar zone using the positions of the rightmost particles. We will return to this in sects. 2.3 and 2.4 below.

Let us recall the definition of the *(domino) height function* associated with a given tiling, [15]. Let $u$ and $v$ be two adjacent lattice points (vertices), in the basic coordinate system, such that the edge connecting them is not covered by a domino. If the edge from $u$ to $v$ has a black square to its left, $h(v) = h(u) + 1$, and if it has a white square to its left, $h(v) = h(u) - 1$. Note that the value of the height function is uniquely determined apart from an overall additive constant. We can fix it by requiring $h(n, 0) = 0$. If $u$ and $v$ are two adjacent lattice points, then $|h(u) - h(v)| = 3$ if the edge is covered by a domino, otherwise $|h(u) - h(v)| = 1$. From this it follows that $h(Q_0^r) = 2n - (2r - 1)$, $h(Q_{n+1}^r) = 2r - 1$ and

$$h(Q_k^r) - h(Q_{k+1}^r) = \begin{cases} -2, & \text{if ES-step} \\ 2, & \text{if SE-step.} \end{cases} \tag{2.3}$$

Consequently, we can use the zig-zag configurations to determine the height at a given point. From (2.3) it follows that there are exactly $r$ ES-steps in $Z_r(\tau)$.

2.2. **The Krawtckouk ensemble.** The *Krawtchouk ensemble*, [38], is a probability measure on $\{0, \dots, K\}^N$ defined by

$$\mathbb{P}_{\mathrm{Kr},N,K,p}[h] = \frac{1}{Z_{N,K,p}} \Delta_N^2(h) \prod_{j=1}^{N} \binom{K}{h_j} p^{h_j} q^{K-h_j}, \tag{2.4}$$

where $0 < p < 1$, $q = 1 - p$, $1 \le N \le K$, $h = (h_1, \dots, h_N) \in \{0, \dots, K\}^N$ and

$$Z_{N,K,p} = N! \left( \prod_{j=0}^{N-1} \frac{j!}{(K-j)!} \right) K!^N (pq)^{N(N-1)/2}. \tag{2.5}$$

Set $w(x) = \binom{K}{x} p^x q^{K-x}$, $0 \le x \le K$ and let $\{p_k(x)\}_{k=0}^K$ be the normalized orthogonal polynomials with respect to $w(x)$ on $\{0, \dots, K\}$, i.e.

$$\sum_{x=0}^{K} p_j(x) p_k(x) w(x) = \delta_{jk}. \tag{2.6}$$

These are multiples of the ordinary Krawtchouk polynomials, [54], and have the integral representation

$$p_n(x) = \binom{K}{x}^{-1/2} (pq)^{-n/2} \frac{1}{2\pi i} \int_\gamma \frac{(1+qz)^x (1-pz)^{K-x}}{z^n} \frac{dz}{z}, \tag{2.7}$$

where $\gamma$ is a circle centered at the origin with radius $\le \min(1/p, 1/q)$. The measure (2.4) has determinantal correlation functions, [52], [67],

$$\det(K_{\mathrm{Kr},N,K,p}(x_i, x_j))_{i,j=1}^m \tag{2.8}$$

where the *Krawtchouk kernel* is given by

$$\begin{aligned} K_{\mathrm{Kr},N,K,p}(x, y) &= \sum_{n=0}^{N-1} p_n(x) p_n(y) (w(x)w(y))^{1/2} \\ &= \frac{\kappa_{N-1}}{\kappa_N} \frac{p_N(x)p_{N-1}(y) - p_{N-1}(x)p_N(y)}{x - y} (w(x)w(y))^{1/2} \end{aligned} \tag{2.9}$$

where $\kappa_n = (n!)^{-1}\binom{K}{n}^{-1/2}(pq)^{-n/2}$ is the leading coefficient in $p_n(x)$.

We will now prove, using the DR-paths, that the measure on the zig-zag configurations induced by the probability measure (2.1) on the tilings is the Krawtchouk ensemble. In the special case of uniform distribution on the set of tilings this was proved in [38] using formulas from [15].

**Theorem 2.2.** *Fix $r$, $1 \le r \le n$, and let $h = (h_1, \dots, h_r)$, where $0 \le h_1 < \dots < h_r \le n$ be given. Then,*

$$\mathbb{P}[p(Z_r(\tau)) = h] = r!\mathbb{P}_{Kr,N,K,q}[h], \tag{2.10}$$

*where $q = w^2(1+w^2)^{-1}$. Hence, if we disregard the ordering of the particles in the zig-zag particle configuration, the probability of $h$ is exactly $\mathbb{P}_{Kr,N,K,q}[h]$.*

*Proof.* By lemma 2.1 we have type I DR-paths from $(r+1-j, 0)$ to $(r, h_j)$, $1 \le j \le r$, in CS-I, and type II DR-paths from $(j, 0)$ to $(n+1-r, n-k_j)$, $1 \le j \le n+1-r$ in CS-II, where $k_1 < \dots < k_{n+1-r}$ and (2.2) holds. Together these describe the whole domino tiling. Let $\omega[h]$ be the weight of all the type I DR-paths between the specified points and $\omega'[h]$ the weight of all the type II DR-paths between the given points. Then,

$$\mathbb{P}[p(Z_r(\tau)) = h] = \frac{\omega[h]\omega'[h]}{\sum_{0\le h_1<\dots<h_r\le n} \omega[h]\omega'[h]}. \tag{2.11}$$

The quantities $\omega[h]$ and $\omega'[h]$ can be computed using the Lindström-Gessel-Viennot method, [51], [27], see also [65], which is a development of the Karlin-McGregor result in a discrete setting. We want to compute the weight of a path that takes $n$ steps to the right and $m$ steps up. Let $a$ be the number of $(1,0)$ steps, $b$ the number of $(0,1)$ steps and $c$ the number of $(1,1)$ steps. Then, $n = a + c$ and $m = b + c$. The number of paths with a given number of steps $a, b, c$ equals

$$\frac{(a+b+c)!}{a!b!c!} = \frac{(n+m-c)!}{(n-c)!(m-c)!c!}.$$

Note that $c$ can take all values between 0 and $\min(n,m)$. The total weight of all possible paths from $(0,0)$ to $(n,m)$ is thus

$$w(n,m) = \sum_{c=0}^{\min(n,m)} \frac{(n+m-c)!}{(n-c)!(m-c)!c!} w^{n+m-2c}. \tag{2.12}$$

If we use the Pochhammer symbol $(a)_k = a(a+1)\dots(a+k-1)$, $(a)_0 = 1$, then

$$w(n,m) = \frac{w^{n+m}}{n!} \sum_{c=0}^{\infty} (m-c+1)_m (n-c+1)_c \frac{w^{-2c}}{c!}. \tag{2.13}$$

The Lindström-Gessel-Viennot method now shows that the total wight of all possible $r$ non-intersecting paths from $(r+1-j, 0)$ to $(r, h_j)$, $1 \le j \le r$, is

$$\omega[h] = \det(w(i-1, h_j))_{i,j=1}^{r}. \tag{2.14}$$

Similarly, the total weight of all possible $n+1-r$ non-intersecting paths from $(j,0)$ to $(n+1-r, n-k_j)$ is

$$\begin{aligned}\omega'[h] &= \det(w(n+1-r-i, n-k_j))_{i,j=1}^{n+1-r} \\ &= \det(w(i-1, k_{n+2-r-j}))_{i,j=1}^{n+1-r}.\end{aligned}$$

If we set $r' = n + 1 - r$, $h'_j = n - k_{n+2-r-j}$, then

$$\omega'[h] = \det(w(i-1, h'_j))_{i,j=1}^{r'}, \tag{2.15}$$

which has exactly the same form as (2.14). This is the advantage of using both types of DR-paths.

Now,

$$\det(w(i-1, h_j))_{i,j=1}^{r} = \left(\prod_{j=1}^{r} \frac{(1+w^2)^{j-1}}{w^{j-1}(j-1)!}\right) \Delta_r(h) \prod_{j=1}^{r} w^{x_j}. \tag{2.16}$$

To see this, insert (2.13) into the left hand side of (2.16), and use the multilinearity of the determinant to obtain

$$\begin{aligned}&\left(\prod_{j=1}^{r} \frac{w^{j-1+h_j}}{(j-1)!}\right) \sum_{c_1,\dots,c_r=0}^{\infty} \prod_{i=1}(i-c_i)_{c_i} \frac{1}{c_i! w^{2c_i}} \det((h_j - c_i + 1)_{i-1})_{i,j=1}^{r} \\ &= \left(\prod_{j=1}^{r} \frac{w^{j-1+h_j}}{(j-1)!}\right) \Delta_r(h) \prod_{i=1}^{r} \sum_{c=0}^{\infty} (i-c)_c \frac{1}{c! w^{2c}},\end{aligned}$$

which equals the right hand side of (2.16) since,

$$\sum_{c=0}^{\infty} (i-c)_c \frac{1}{c! w^{2c}} = \sum_{c=0}^{i-1} \binom{i-1}{c} \frac{1}{w^{2c}} = (1 + \frac{1}{w^2})^{i-1}.$$

From (2.15) we obtain, after some manipulation,

$$\omega'[h] = \left(\prod_{j=1}^{n+1-r} \frac{(1+w^2)^{j-1}}{w^{j-1}(j-1)!}\right) \Delta_{n+1-r}(k) \prod_{j=1}^{n+1-r} w^{n-k_j}.$$

Lemma 2.2 in [38] shows that if $s_1 < \cdots < s_N$ and $r_1 < \cdots < r_M$ and the union of these two sets of numbers is exactly $\{0, \dots, N+M-1\}$, then

$$\Delta_M(r) = \left(\prod_{j=1}^{N+M-1} j!\right) \left(\prod_{j=1}^{N} \frac{1}{s_j!(N+M-1-s_j)!}\right) \Delta_N(s). \tag{2.17}$$

If we use this formula, we obtain

$$\omega[h]\omega'[h] = w^{-r(r-1)}(1+w^2)^{n(n+1)/2 - nr + r(r-1)} \prod_{j=1}^{r-1} \frac{(n-j)!}{n! j!} \Delta_r(h)^2 \prod_{j=1}^{r} \binom{n}{h_j} w^{2h_j}.$$

By this formula and (2.5) we obtain

$$\omega[h]\omega'[h] = (1+w^2)^{n(n+1)/2} \frac{r!}{Z_{r,n,q}} \Delta_r(h)^2 \prod_{j=1}^{r} \binom{n}{h_j} q^{h_j} p^{n-h_j},$$

where $q = w^2(1+w^2)^{-1}$. Hence

$$\sum_{0 \le h_1 < \cdots < h_r \le n} \omega[h]\omega'[h] = (1+w^2)^{n(n+1)/2}, \tag{2.18}$$

and the theorem follows from (2.11). $\square$

As a corollary we obtain the following result first proved in [15], see also [36].

**Corollary 2.3.** *The number of elements in $\mathcal{T}(A_n)$ is $2^{n(n+1)/2}$, and the probability of having $2k$ vertical tiles, $0 \le k \le n(n+1)/2$, is*

$$\binom{n(n+1)/2}{k}\left(\frac{w^2}{1+w^2}\right)^k\left(\frac{1}{1+w^2}\right)^{n(n+1)/2-k}. \tag{2.19}$$

*The probability of having an odd number of vertical tiles is zero.*

*Proof.* The first result follows by putting $w = 1$ in (2.18), and (2.19) follows by expanding the right hand side of (2.18) using the binomial theorem. □

2.3. **Asymptotics in the Krawtchouk ensemble.** There is an *equilibrium measure* associated with the Krawtchouk ensemble, see sect. 2.2. in [37], and sect. 4.2 below for some more details. If $r \to \infty$ and $n \to \infty$ in such a way that $r/n \to t \in (0,1)$, then the expectation of the discrete measure $\frac{1}{r}\sum_{j=1}^{r}\delta_{h_j/n}$ converges weakly to the equilibrium measure $u_{t,q}(x)dx$, i.e. the equilibrium measure gives the asymptotic distribution of the particles. The equilibrium distribution is scaled so that its support is contained in $[0,1]$. The equilibrium measure also gives the asymptotic distribution of the zeroes of the Krawtchouk polynomials scaled to $[0,1]$. See [14] for this result and explicit formulas for the equilibrium measure and its support. Since the position of the rightmost particle determines the boundary of the north polar zone (and we can make an analogous analysis for the other polar zones or use symmetry), we can use the equilibrium measure to prove the arctic ellipse theorem, [36] and [10]. The arctic ellipse theorem of Jockush, Propp and Shor says that the boundary of the temperate zone, scaled by $1/n$, converges in probability to an ellipse as $n \to \infty$.

**Theorem 2.4.** *If we scale the Aztec diamond by $1/n$ in the original coordinate system, then the boundary $\partial T_n$ of the temperate zone in a rescaled andom tiling of $A_n$ under the probability measure (2.1), converges in probability as $n \to \infty$ , $r/n \to t \in (0,1)$, to the ellipse $E$,*

$$\frac{x^2}{p} + \frac{y^2}{q} = 1, \tag{2.20}$$

*in the sense that $\mathbb{P}[dist(\partial T_n, E) \ge \epsilon] \to 0$ for any fixed $\epsilon > 0$. Let $dist_I(\partial T_n, E)$ be the maximal distance from a point on $\partial T_n$ inside $E$ to $E$, and $dist_O(\partial T_n, E)$ be the same thing but from a point outside $E$. Given $\epsilon > 0$, there are positive constants $I(\epsilon)$ and $J(\epsilon)$ such that*

$$\limsup_{n\to\infty}\frac{1}{n^2}\log\mathbb{P}[dist_I(\partial T_n, E) \ge \epsilon] \le -I(\epsilon)$$

*and*

$$\limsup_{n\to\infty}\frac{1}{n}\log\mathbb{P}[dist_O(\partial T_n, E) \ge \epsilon] \le -J(\epsilon)$$

*Proof.* We will indicate how the shape (2.20) is obtained. The large deviation formulas, which imply the convergence in probability to the ellipse, follow from theorem 2.2 in [37]. See sect. 4.2 below for some more details in the analogous result for rhombus tilings of a hexagon. Let $(x_I, y_I)$ be coordinates in CS-I and $(x, y)$ coordinates in the original coordinate system. Then, $x = n + 1/2 - x_I - y_I$, $y = 1/2 - x_I + y_I$. It follows from [14] that the right endpoint of the support of the

equilibrium measure is given by $\beta(t,q) = tp+(1-t)q+2\sqrt{t(1-t)pq}$ if $0 < t \le 1-q$. The connection between the position of the rightmost particle and the boundary of the north polar zone described above imply that the limiting boundary of the north polar zone must be the curve $(0, 1-q] \ni t \to (t, \beta(t,q)) = (x_I, y_I)$ in CS-I. Using the coordinate transformation we find that, in the original coordinate system, points on this curve satisfy (2.20). □

Let $\nu[a,b](h)$ be the number of particles in the interval $[a,b]$ in the particle configuration $h$. From the formula (2.3) and $h(Q^r_{n+1}) = 2r-1$, we obtain

$$\begin{aligned} h(Q^r_k) - (2r-1) &= \sum_{j=k}^{n} [h(Q^r_j) - h(Q^r_{j+1})] \\ &= -2\nu[0, n-k] + 2(n-k+1-\nu[0,n-k]) = 2(n-k+1) - 4\nu[0,n-k]. \end{aligned}$$

Consequently,

$$h(Q^r_k) = 2(n-k+r) + 1 - 4\nu[0,n-k]. \tag{2.21}$$

Assume that $k/n \to \tau$ and $r/n \to t$, $0 < \tau < t$ as $n \to \infty$. Then, by (2.18) and the weak convergence of the particle distribution,

$$\lim_{n\to\infty} \frac{h(Q^r_k)}{n} = 2(1-\tau+t) - 4t\int_0^{1-\tau} u_{t,q}(x)dx. \tag{2.22}$$

The precise form of the equilibrium measure is given in [14] and using this we can work out the asymptotic height function. We will not evaluate these integrals here. We can also obtain large deviation formulas (and estimates, compare lemma 4.1 in [37]) for macroscopic deviations from the asymptotic (average) height function. This is analogous to the large deviation formulas for the Wigner semi-circle law, [4]. We will not develop the details, since it is very analogous to the corresponding random matrix results. See [10] for previous large deviation estimates and asymptotics for the height function.

We will now analyze the fluctuations of the height function, or what is equivalent, by (2.21), the fluctuations in the number of particles in an interval in the Krawtchouk ensemble (2.4). Let $I = \{b-L, b-L+1, \dots, b\} \subseteq \{0, \dots, K\}$ be an "interval" of length $L$ and let $\nu(I)$ denote the number of particles in $I$,

$$\nu(I) = \#\{h_i\,;\, 1 \le i \le N \text{ and } h_i \in I\},$$

We want to prove the following result for the variance of $\nu(I)$, the *number variance*.

**Proposition 2.5.** *Assume that $N/K \to t \in (0, 1/2]$, $p = q = 1/2$, $b/K \to \beta$, $(b-L)/K \to \beta' \le \beta$ as $N, K, L \to \infty$, where $\beta$ or $\beta'$ belongs to the interior, $S$, of the support of the equilibrium measure $u_{t,1/2}$. Then*

$$\lim \frac{var(\nu(I))}{\log L} = \frac{1}{\pi^2}\xi_{\beta,\beta'}, \tag{2.23}$$

*where $\xi_{\beta,\beta'}$ is $= 1$ if both $\beta$ and $\beta'$ belong to $S$ and $= 1/2$ otherwise.*

This type of result was conjectured in [59]. We will give the proof of this proposition below, which is rather long. The case $p \ne q$ could also be worked out, but we stick with $p = q = 1/2$ for simplicity. Once we have this result we can apply the Costin/Lebowtiz/Soshnikov argument, [11], [61], to prove that the fluctuations are normal. This type of results have been proved in other tiling models by Kenyon, see [45], [46], [47].

**Theorem 2.6.** *With the same assumptions as in the proposition,*

$$\frac{\nu(I)-\mathbb{E}[\nu(I)]}{\sqrt{var(I)}} \Rightarrow N(0,1) \tag{2.24}$$

*as $K\to\infty$. i.e. we have convergence in distribution to a standard normal random variable.*

*Proof.* By a theorem in [61], p. 8, see also [62], the result (2.24) follows from (2.23) since we have a determinantal random point field, i.e. the correlation functions are given by determinants as in (2.8). The kernel (2.9) defines a trace class operator (it has finite rank) $\mathcal{K}$, which satisfies $0\le\mathcal{K}\le I$. This follows from (2.9) and the orthogonality (2.6). Hence the conditions in the Costin-Lebowitz-Soshnikov theorem are satisfied. $\square$

Combining this theorem with (2.21) we obtain the next theorem.

**Theorem 2.7.** *Take the uniform distribution on $\mathcal{T}(A_n)$ and $0\le r\le n/2$ (the case $n/2\le r\le n$ is similar by symmetry). Let $Q_k^r=(-r+k,n+1-k-r)$ as before and let $h(Q_k^r)$ be the value of the domino height function above this point. If $r/n\to t\in(0,1)$, $k/n\to\kappa\in(1/2-\sqrt{t(1-t)},1/2+\sqrt{t(1-t)})\doteq U_t$ or $j/n\to\kappa'\in U_t$, and $|k-j|\to\infty$, then*

$$\frac{h(Q_k^r)-h(Q_j^r)-\mathbb{E}[h(Q_k^r)-h(Q_j^r)]}{4\sqrt{\xi_{\kappa,\kappa'}\pi^{-2}\log|k-j|}} \Rightarrow N(0,1). \tag{2.25}$$

*Here $\xi_{\kappa,\kappa'}=1$ if $\kappa,\kappa'\in U_t$ and $\xi_{\kappa,\kappa'}=1/2$ if one of $\kappa$ or $\kappa'$ does not belong to $U_t$.*

Note that the condition on $\kappa,\kappa'$ corresponds exactly to the condition that one of $\beta$ or $\beta'$ in proposition 2.5 belongs to the interior of the support of the equilibrium measure.

We turn now to the proof of proposition 2.5 which is rather lengthy. The proof is based on sufficiently good asymptotic control of the Krawtchouk kernel, (2.9). For results on asymptotics of Krawtchouk polynomials see [35]. We state the needed result as a lemma, which we will prove later. Let $\rho(\xi)$ be defined by (2.58) below. We consider the case when $p=1/2$ and $N/K\to t$. Then, $\rho'(\xi)=u_{t,1/2}(\xi)$ is the equilibrium measure, which is suppported in $[1/2-\sqrt{t(1-t)},1/2+\sqrt{t(1-t)}]$, see [14]. A computation gives,

$$\rho'(\xi)=\frac{1}{\pi}\arctan\frac{\sqrt{t(1-t)-(\xi-1/2)^2}}{\sqrt{1/4-t(1-t)}}. \tag{2.26}$$

**Lemma 2.8.** *Consider the Krawtchouk kernel (2.9) with $p=1/2$ and let $\delta>0$ (small). Set $t=N/K$ and assume $0<t\le 1/2$. If $|x/K-1/2|\le\sqrt{t(1-t)}-\delta$ and $|y/K-1/2|\le\sqrt{t(1-t)}-\delta$, then there is a constant $C$, independent of $N,K,x$ and $y$ such that*

$$\left|(x-y)K_{Kr,N,K,1/2}(x,y)-\frac{\sin\pi K(\rho(x/K)-\rho(y/K))}{\pi}\right|\le C\left(\frac{1}{\sqrt{K}}+\frac{|x-y|}{K}\right). \tag{2.27}$$

*Also, if $|x/K-1/2|\le\sqrt{t(1-t)}-\delta$, there is a constant $C$ such that for $y\ge x$,*

$$|(x-y)K_{Kr,N,K,1/2}(x,y)|\le C\min(K^{1/4},|t(1-t)-(y/K-1/2)^2|^{-1/2})\frac{K^{1/4}}{(K-y)^{1/4}}. \tag{2.28}$$

*There is also an analogous result for $y \leq x$. If $1/2 + \sqrt{t(1-t)} < 1$, we can obtain a much better estimate for $y$ outside the support of the equilibrium measure; compare (2.66) below.*

We will now prove proposition 2.5.

*Proof.* Write $\mathcal{K}(x,y) = K_{\mathrm{Kr},N,K,1/2}(x,y)$. It follows from (2.9) and the orthogonality (2.6), that $\mathcal{K}(x,y)$ is a reproducing kernel

$$\sum_{j=0}^{K} \mathcal{K}(i,j)\mathcal{K}(j,k) = \mathcal{K}(i,k). \tag{2.29}$$

Set $\tilde{I} = \{0,\dots,K\} \setminus I$. Then by the formulas (2.8) and (2.29), we see that

$$\begin{aligned} \operatorname{var}[\nu(I)] &= \sum_{j\in I} \mathcal{K}(j,j) - \sum_{i,j\in I} \mathcal{K}(i,j)^2 \\ &= \sum_{j\in I}\left(\sum_{i=0}^{K} \mathcal{K}(j,i)\mathcal{K}(i,j)\right) - \sum_{i,j\in I} \mathcal{K}(i,j)^2 \\ &= \sum_{j\in I}\sum_{i\in\tilde{I}} \mathcal{K}(i,j)^2 = \Sigma_1 + \Sigma_2, \end{aligned} \tag{2.30}$$

where

$$\begin{aligned} \Sigma_1 &= \sum_{i=0}^{L}\sum_{j=1}^{K-b} \mathcal{K}(b-i,b+j)^2 \\ \Sigma_1 &= \sum_{i=0}^{b-L}\sum_{j=1}^{L} \mathcal{K}(b-L-i,b-L+j)^2. \end{aligned}$$

We will consider the case when $\beta$ lies in the support $S$ of the equilibrium measure. The contribution to $\Sigma_1$ comes from the right endpoint of the interval, and the contribution to $\Sigma_2$ from the left endpoint. We will show that

$$\Sigma_1 = \frac{1}{2\pi^2}\log L + O(\log(\log L)). \tag{2.31}$$

The same thing is true for $\Sigma_2$, with an analogous proof, if $\beta' \in S$, so if we establish (2.31) the proposition is proved.

Let $a(L) = [L/\log L]$. We split $\Sigma_1$ into two parts

$$\begin{aligned} \Sigma_1 &= \sum_{i=0}^{L}\sum_{j=1}^{a(L)} \mathcal{K}(b-i,b+j)^2 + \sum_{i=0}^{L}\sum_{j=a(L)+1}^{K-b} \mathcal{K}(b-i,b+j)^2 \\ &= \Sigma_1' + \Sigma_1''. \end{aligned} \tag{2.32}$$

Now, by (2.28),

$$\Sigma_1'' \le \sum_{i=0}^{L} \sum_{j=a(L)+1}^{[(\alpha-\delta/2)K]-b} \frac{CK^{1/2}}{|K\alpha-(b+j)|^{1/2}} \frac{1}{(i+j)^2}$$
$$+\sum_{i=0}^{\infty} \sum_{j=[(\alpha-\delta/2)K]-b}^{[\alpha K]-[K^{1/2}]-b} \frac{CK^{1/2}}{|K\alpha-(b+j)|^{1/2}} \frac{1}{(i+j)^2}$$
$$+\sum_{i=0}^{\infty} \sum_{j=[\alpha K]-[K^{1/2}]-b}^{[\alpha K]-[K^{1/2}]+b} \frac{CK^{1/4}}{(i+j)^2}$$
$$+\sum_{i=0}^{\infty} \sum_{j=[\alpha K]-[K^{1/2}]+b}^{K-b} \frac{CK^{1/2}}{|K\alpha-(b+j)|^{1/2}} \frac{1}{(i+j)^2},$$

where $\alpha = 1/2+\sqrt{t(1-t)}$, the right endpoint of the support. The first sum is

$$\le C\sum_{i=0}^{L} \sum_{j=a(L)+1}^{\infty} \frac{1}{(i+j)^2} \le C\log(\log L).$$

In the last three sums, we use $\sum_{i=0}^{\infty} 1/(i+j)^2 \le 1/(j-1)$, and it is then easy to see that the $j$-sums are $\le C$. Thus

$$\Sigma_1'' \le C(1+\log(\log L)). \tag{2.33}$$

We also split $\Sigma_1'$ into two sums

$$\Sigma_1' = \sum_{i=0}^{a(L)} \sum_{j=1}^{a(L)} \mathcal{K}(b-i,b+j)^2 + \sum_{i=a(L)+1}^{L} \sum_{j=1}^{a(L)} \mathcal{K}(b-i,b+j)^2$$
$$= S_1+S_2. \tag{2.34}$$

The second sum is estimated in the same way as $\Sigma_1''$ using (2.28), and this gives

$$S_2 \le C. \tag{2.35}$$

To control $S_1$ we use (2.27), which gives

$$\left| S_1 - \sum_{i=0}^{a(L)} \sum_{j=1}^{a(L)} \frac{\sin^2 \pi K(\rho((b-i)/K)-\rho((b+j)/K))}{\pi^2(i+j)^2} \right|$$
$$\le C\sum_{i=0}^{a(L)} \sum_{j=1}^{a(L)} \left(\frac{1}{\sqrt{K}}+\frac{i+j}{K}\right)\frac{1}{(i+j)^2} \le C. \tag{2.36}$$

If we use $\sin^2 x = (1+\cos 2x)/2$ and observe that

$$\sum_{i=0}^{a(L)} \sum_{j=1}^{a(L)} \frac{1}{(i+j)^2} = \log L + O(\log(\log L)), \tag{2.37}$$

we see that what remains to be proved is

$$\left| \sum_{i=0}^{a(L)} \sum_{j=1}^{a(L)} \frac{\cos 2\pi K(\rho((b-i)/K)-\rho((b+j)/K))}{\pi^2(i+j)^2} \right| \le C\log(\log L). \tag{2.38}$$

When we have this we note that (2.32) - (2.38) imply (2.31) and we are done.

To prove (2.38) we use summation by parts. Set $S(0) = 0$ and

$$S(j) = \sum_{n=1}^{j} \exp(-2\pi K i \rho(\frac{b+n}{K})), \quad 1 \le j \le a(L).$$

Consider the expression

$$\sum_{m=0}^{a(L)} e^{2\pi K i \rho(\frac{b-m}{K})} \sum_{j=1}^{a(L)} \frac{1}{(m+j)^2} e^{-2\pi K i \rho(\frac{b+j}{K})}, \tag{2.39}$$

whose real part is what we want. Summation by parts gives

$$\sum_{j=1}^{a(L)} \frac{1}{(m+j)^2} e^{-2\pi K i \rho(\frac{b+j}{K})} = \frac{1}{(m+a(L))^2} S(a(L)) + \sum_{j=1}^{a(L)-1} \frac{2(m+j)+1}{(m+j)^2(m+j+1)^2} S(j). \tag{2.40}$$

Since $|S(a(L)| \le a(L)$, the first term in (2.40) gives a contribution $\le 1$ to the expression (2.39). The contribution of the second term in (2.40) to (2.39) is

$$\le \sum_{m=0}^{a(L)} \sum_{j=1}^{a(L)-1} \frac{2(m+j)+1}{(m+j)^2(m+j+1)^2} S(j). \tag{2.41}$$

Fix an integer $\Delta > 1$ and assume that $\Delta < j \le a(L)$. Write $j = k\Delta + r$, where $0 \le r < \Delta$, and $k \le j/\Delta$. Then

$$S(j) = \sum_{u=1}^{k} \sum_{v=1}^{\Delta} e^{-2\pi K i \rho(\frac{b+u\Delta+v}{K})} + \sum_{v=1}^{r} e^{-2\pi K i \rho(\frac{b+k\Delta+v}{K})}. \tag{2.42}$$

Write $\xi = (b + u\Delta)/K$. Then $\rho(\frac{b+u\Delta+v}{K}) = \rho(\xi) + \frac{v}{K}\rho'(\xi) + O(\frac{v^3}{K^2})$. Note that $\rho'(\xi) > 0$ since we are inside the support of the equilibrium measure $\rho'$. Also, $\rho'(\xi) < 1$ by (2.26). Using this we see that

$$\sum_{v=1}^{\Delta} \left| e^{-2\pi K i \rho(\xi + \frac{v}{K})} - e^{-2\pi K i \rho(\xi) - 2\pi i v \rho'(\xi)} \right| \le C \sum_{v=1}^{\Delta} \frac{v^2}{K} \le C \frac{\Delta^3}{K}. \tag{2.43}$$

Now,

$$\left| \sum_{v=1}^{\Delta} e^{-2\pi K i \rho(\xi) - 2\pi i v \rho'(\xi)} \right| \le C, \tag{2.44}$$

since $0 < \rho'(\xi) < 1$. Similarly

$$\left| \sum_{v=1}^{\Delta} e^{-2\pi K i \rho(\frac{b+k\Delta+v}{K})} \right| \le C(1 + \frac{\Delta^3}{K}). \tag{2.45}$$

Combining (2.42) to (2.45) we obtain

$$|S(j)| \le Ck(1 + \frac{\Delta^3}{K}) \le Cj(\frac{1}{\Delta} + \frac{\Delta^2}{K})$$

for $\Delta < j \le a(L)$. Since evidently $|S(j)| \le j$ for $1 \le j \le \Delta$, we obtain the estimate

$$\left| \sum_{m=0}^{a(L)} \sum_{j=1}^{a(L)-1} \frac{2(m+j)+1}{(m+j)^2(m+j+1)^2} S(j) \right| \le C \log \Delta + C(\frac{1}{\Delta} + \frac{\Delta^2}{K}) \log L$$

of the expression in (2.41). If we take $\Delta = [\log L]$, we see that the absolute value of the expression (2.39) is $\le \log(\log L)$, which gives the bound (2.38). $\square$

We still have to prove lemma 2.8.

*Proof.* Let $p = q = 1/2$ in (2.9) and make the change of variable $z = 2w$. This gives,

$$p_n(z) = \binom{K}{n}^{-1/2} \frac{1}{2\pi i} \int_\gamma \frac{(1+w)^x(1-w)^{K-x}}{w^n} \frac{dw}{w}.$$

Set

$$A_N(x,y) = \frac{\kappa_{N-1}}{\kappa_N}(w(x)w(y))^{1/2} \binom{K}{N}^{-1/2} \binom{K}{N-1}^{-1/2}, \tag{2.46}$$

$G_N(z;x) = (1+z)^x(1-z)^{K-x}w^{-N}$ and $L_N(x,y) = (x-y)K_{\mathrm{Kr}\,,N,K,1/2}(x,y)$. Then,

$$L_N(x,y) = A_N(x,y) \int_\gamma \frac{dz}{2\pi i z} \int_\gamma \frac{dw}{2\pi i w} G_N(z;x)G_N(w;y)(w-z).$$

Set $\xi = x/K$ and $\eta = y/K$ and define $f(z) = \xi \log(1+z) + (1-\xi)\log(1-z) - t\log z$, which is the relevant function in the saddle point argument. The equation $f'(z) = 0$ has the solutions $z_c^\pm = (1-t)^{-1}(\xi - 1/2 \pm i\sqrt{t(1-t) - (\xi-1/2)^2})$. Let $r_c = |z_c^\pm| = \sqrt{t(1-t)^{-1}}$. We choose $\gamma$ to be the circle with radius $r_c$, and write $z_c^\pm = r_c \exp(\pm i\theta_c(\xi))$, where

$$\cos\theta_c(\xi) = \frac{\xi - 1/2}{\sqrt{t(1-t)}}, \quad 0 \le \theta_c \le \pi. \tag{2.47}$$

We assume that $\xi \ge 1/2$ (the other case is similar by symmetry). Also, we will first assume that $\xi \le 1/2 + \sqrt{t(1-t)}$, i.e. we are inside the support of the equilibrium measure. It follows that $0 \le \theta_c \le \pi/2$. We obtain,

(2.48)
$$L_N(x,y) = A_N(x,y)\frac{r_c}{(2\pi)^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} G_N(r_c e^{i\theta};x)G_N(r_c e^{i\phi};y)(e^{i\phi} - e^{i\theta})d\theta d\phi,$$

and this is the formula we will use. Let

$$\begin{aligned} g(\theta) = g(\theta;x) &= \log|G_N(r_c e^{i\theta};x)| \\ &= -N\log r_c + \frac{x}{2}\log|1 + r_c e^{i\theta}|^2 + \frac{K-x}{2}\log|1 - r_c e^{i\theta}|^2. \end{aligned}$$

Taking the derivative gives

$$g'(\theta) = \frac{2Kt}{1-t}\sin\theta \left[ \frac{\cos\theta - \frac{\xi-1/2}{\sqrt{t(1-t)}}}{|1 + r_c e^{i\theta}|^2 |1 - r_c e^{i\theta}|^2} \right]. \tag{2.49}$$

From this formula we see that

$$g(\theta) \le g(\theta_c), \quad -\pi \le \theta \le \pi. \tag{2.50}$$

Taking the absolute values in (2.48) gives

$$\begin{aligned}|L_N(x,y)| \le {} & \frac{2r_c}{\pi^2}A_N(x,y)e^{g(\theta_c(\xi);x)+g(\theta_c(\eta);y)} \\ & \times \left(\int_0^\pi e^{g(\theta;x)-g(\theta_c(\xi);x)}d\theta\right)\left(\int_0^\pi e^{g(\theta;y)-g(\theta_c(\eta);y)}d\theta\right). \end{aligned} \tag{2.51}$$

Fix $\delta > 0$ (small). If $\delta \le \theta_c \le \pi/2$ we can make a quadratic approximation around $\theta_c$ to obtain the estimate

$$\int_0^\pi e^{g(\theta)-g(\theta_c)}d\theta \le \frac{C}{\sqrt{K}}, \tag{2.52}$$

for some constant $C$. If $\theta_c$ is close to 0 we have to be more careful. A computation using (2.49) shows that there is a constant $\alpha > 0$ such that

$$\int_0^\pi e^{g(\theta)-g(\theta_c)}d\theta \le \int_0^\infty e^{-\alpha K(u^2-u_c^2)^2}du. \tag{2.53}$$

Here we have taken $u = \sin\theta$, $u_c = \sin\theta_c$. Now,

$$\int_0^\infty e^{-\alpha K(u^2-u_c^2)^2}du \le C\min(\frac{1}{u_c\sqrt{K}}, \frac{1}{K^{1/4}}) \tag{2.54}$$

and hence

$$\int_0^\pi e^{g(\theta;x)-g(\theta_c(\xi);x)}d\theta \le C\min(\frac{1}{\sin\theta_c(\xi)\sqrt{K}}, \frac{1}{K^{1/4}}). \tag{2.55}$$

To prove (2.54) assume that $u_c \ge 1/K^{1/4}$. Set $u = u_c(1+s)$. The left hand side becomes

$$u_c\int_{-1}^\infty e^{-\alpha K u_c^4 s^2(s+2)^2}ds \le u_c\int_{-\infty}^\infty e^{-\alpha K u_c^4 s^2}ds = \frac{C}{u_c\sqrt{K}}.$$

On the other hand if $u_c \le 1/K^{1/4}$, then the left hand side equals

$$\int_0^{u_c\sqrt{2}} e^{-\alpha K(u^2-u_c^2)^2}du + \int_{u_c\sqrt{2}}^\infty e^{-\alpha K(u^2-u_c^2)^2}d \le \sqrt{2}u_c + \int_0^\infty e^{-\alpha K u^4}du \le \frac{C}{K^{1/4}},$$

and we have proved (2.54). Note that $\sin\theta_c(\xi) = \frac{1}{\sqrt{t(1-t)}}\sqrt{t(1-t)-(\xi-1/2)^2}$, so (2.55) can be written

$$\int_0^\pi e^{g(\theta;x)-g(\theta_c(\xi);x)}d\theta \le C\min(\frac{1}{\sqrt{K}\sqrt{|t(1-t)-(\xi-1/2)^2|}}, \frac{1}{K^{1/4}}). \tag{2.56}$$

If $\xi \ge 1/2 + \sqrt{t(1-t)}$ we can take $\theta_c = 0$ and then (2.56) still holds by a similar argument using (2.49). Thus, for any $0 \le x, y \le K$, by (2.51) and (2.56),

$$\begin{aligned}|L_N(x,y)| \le {} & CA_N^*(x,y)\min(|t(1-t)-(\xi-1/2)^2|^{-1/2}, K^{1/4}) \\ & \times \min(|t(1-t)-(\eta-1/2)^2|^{-1/2}, K^{1/4}), \end{aligned} \tag{2.57}$$

where $\theta_c(\xi)$ is given by (2.47) if $|\xi - 1/2| \le \sqrt{t(1-t)}$, $\theta_c(\xi) = 0$ if $|\xi-1/2| > \sqrt{t(1-t)}$ and

$$A_N^*(x,y) = \frac{1}{K}r_cA_N(x,y)e^{g(\theta_c(\xi);x)+g(\theta_c(\eta);y)}.$$

Let $\delta > 0$ and consider $x, y$ such that $|\xi - 1/2| \le \sqrt{t(1-t)} - \delta$ and $|\eta - 1/2| \le \sqrt{t(1-t)} - \delta$, i.e. we are inside the support of the equilibrium measure. Now,

$$\frac{d^2}{d\theta^2} f(r_c e^{i\theta})\Big|_{\theta=\theta_c} = -(z_c^+) \doteq -d_+(\xi)$$

and a straightforward computation shows that $\Re d_+(\xi) > 0$. Furthermore, if $d_-(\xi) \doteq -\frac{d^2}{d\theta^2} f(r_c e^{i\theta})|_{\theta=-\theta_c}$, then $d_-(\xi) = \overline{d_+(\xi)}$. Also, for the $\xi$ and $\eta$ we are considering we have that $|d_+(\xi) - d_+(\eta)| \le C|\xi - \eta|$. A standard local saddle-point argument now gives, using (2.48), (2.49) and (2.50),

$$L_N(x,y) = \frac{r_c A_N(x,y)}{2\pi}(I_{++} + I_{-+} + I_{+-} + I_{--}),$$

where $(a, b = \pm)$,

$$I_{ab} = \left[\frac{1}{K\sqrt{d_a(\xi)d_b(\eta)}} + O(\frac{1}{K^{3/2}})\right] G_N(r_c e^{ai\theta_c(\xi)}; x) G_N(r_c e^{ai\theta_c(\eta)}; y)(e^{bi\theta_c(\eta)} - e^{ai\theta_c(\xi)}).$$

Write,

$$G_N(r_c e^{ai\theta_c(\xi)}; x) = e^{g(\theta_c(\xi);x) + iK\pi\rho(\xi)}, \tag{2.58}$$

which defines $\rho$ in lemma 2.8. Since $|\exp(i\theta_c(\eta)) - \exp(i\theta_c(\xi))| \le C|\xi - \eta|$, $d_+(\xi) = \overline{d_-(\xi)}$ and $|d_+(\xi) - d_+(\eta)| \le C|\xi - \eta|$, we get

$$\Big| L_N(x,y) - \frac{A_N^*(x,y)}{2\pi|d_+(\xi)|}\Big[e^{iK(\rho(\xi)-\rho(\eta))}(e^{-i\theta_c(\xi)} - e^{i\theta_c(\eta)}) + e^{iK(-\rho(\xi)+\rho(\eta))}(e^{i\theta_c(\xi)} - e^{-i\theta_c(\eta)})\Big]\Big| \le CA_N^*(x,y)(\frac{1}{\sqrt{K}} + |\xi - \eta|),$$

which can be written

$$\left| L_N(x,y) - \frac{2\sin\theta_c(\xi)}{\pi|d_+(\xi)|} A_N^*(x,y) \sin \pi K(\rho(\xi) - \rho(\eta)) \right| \le CA_N^*(x,y)(\frac{1}{\sqrt{K}} + \frac{|x-y|}{K}). \tag{2.59}$$

We now investigate $A_N^*(x,y)$ and start with the case when both $\xi$ and $\eta$ are inside the support of the equilibrium measure. Inserting the formulas for $\kappa_n$ and $w(x)$ in (2.46) we obtain

$$A_N(x,y) = (K - N + 1)\binom{K}{N}^{-1} 2^{-K}\binom{K}{x}^{1/2}\binom{K}{y}^{1/2}. \tag{2.60}$$

Furthermore, a computation shows that

$$\begin{aligned} e^{g(\theta_c(\xi);x)} &= r_c^{-N}(1 + r_c^2 + 2r_c\cos\theta_c)^{x/2}(1 + r_c^2 - 2r_c\cos\theta_c)^{(K-x)/2} \\ &= \frac{(1-t)^{N/2} 2^{K/2}}{t^{N/2}(1-t)^{K/2}K^{K/2}} x^{x/2}(K-x)^{(K-x)/2}. \end{aligned} \tag{2.61}$$

Stirling's formula gives the asymptotic formulas

$$\binom{K}{x} = \frac{K^K}{(K-x)^{K-x}x^x\sqrt{K}} \frac{1}{\sqrt{2\pi\xi(1-\xi)}}(1 + O(\frac{1}{K})) \tag{2.62}$$

and

$$\binom{K}{N} = \frac{1}{(1-t)^{K-N}t^N\sqrt{K}} \frac{1}{\sqrt{2\pi t(1-t)}}(1 + O(\frac{1}{K})). \tag{2.63}$$

Combining (2.60) - (2.63) gives

$$A_N^*(x,y) = \frac{t(1-t)}{2\sqrt{\xi(1-\xi)}}(1+O(\frac{1}{K})+O(|\xi-\eta|)). \tag{2.64}$$

A computation shows that

$$\frac{\sin\theta_c}{|d_+(\xi)|} = \frac{\sqrt{\xi(1-\xi)}}{t(1-t)}, \tag{2.65}$$

and hence (2.27) in the proposition follows by combining (2.59), (2.64) and (2.65).

We also have to estimate $A_N^*(x,y)$ when $\xi$ or $\eta$ is outside the support of the equilibrium measure. Assume that $\xi$ is inside and $\eta$ outside the support (or close to the edge of the support), so that $\theta_c(\eta)=0$. We have that

$$e^{g(0;y)} = r_c^{-N}(1+r_c)^y(1-r_c)^{K-y}.$$

Set

$$B(y) = r_c^{-N}\left(1+r_c^2+2r_c\frac{\eta-1/2}{\sqrt{t(1-t)}}\right)^{y/2}\left(1+r_c^2-2r_c\frac{\eta-1/2}{\sqrt{t(1-t)}}\right)^{(K-y)/2},$$

which corresponds to the expression (2.61) with $x=y$. We want to show that

$$\frac{e^{g(0;y)}}{B(y)} \le \gamma^K \le 1, \tag{2.66}$$

with $\gamma<1$ if $\eta > 1/2+\sqrt{t(1-t)}$. A computation gives

$$\frac{e^{g(0;y)}}{B(y)} = \left(\frac{\alpha}{\eta}\right)^{y/2}\left(\frac{1-\alpha}{1-\eta}\right)^{(K-y)/2} = \left[e^{\eta\log\frac{\alpha}{\eta}+(1-\eta)\log(\frac{1-\alpha}{1-\eta})}\right]^K,$$

where $\alpha = 1/2+\sqrt{t(1-t)}$, $0\le\alpha\le\eta\le 1$. Set $\gamma = e^{\eta\log\frac{\alpha}{\eta}+(1-\eta)\log(\frac{1-\alpha}{1-\eta})}$ and note that $\eta\log\frac{\alpha}{\eta}+(1-\eta)\log(\frac{1-\alpha}{1-\eta}) \le \log(\alpha+1-\alpha)=0$ by convexity; if $\eta>\alpha$ we get a strict inequality. When $y$ is close to $K$ we have to be somewhat more careful in estimating the binomial coefficients. We use

$$\binom{K}{y} = \frac{K^K}{(K-y)^{K-y}y^y}\sqrt{\frac{K}{y(K-y)}}(1+O(\frac{1}{y})+O(\frac{1}{K-y})).$$

Using this formula and proceeding as before we obtain

$$A_N^*(x,y) = \frac{t(1-t)}{(\xi(1-\xi))^{1/4}}\frac{K^{1/4}}{(K-y)^{1/4}}(1+O(\frac{1}{|K-y|})+O(\frac{1}{K})), \tag{2.67}$$

instead of (2.64). Combining (2.57) and (2.67) we obtain (2.28) and the proposition is proved. Note that when $t=1/2$ some modifications in the arguments above are needed. We will omit the details. $\square$

2.4. **The corner growth model.** We can draw the type I DR-paths in a different way so that they look like the heights of a cascade of discrete polynuclear growth (PNG) models. If we place a W-domino so that it has corners at $(0,0),(1,0),(1,2)$ and $(0,2)$ we draw a path by connecting the points $(0,1/2)$, $(1/2,1/2)$, $(1/2,3/2)$ and $(1,3/2)$ with straight line segments. We draw a path on an E-domino analogously, it is the mirror image of the W-domino in the middle vertical line. In this way, from the DR-paths, we obtain height curves, which we can think of as graphs of functions $h_k(x,n)$, $1\le k\le n$, where the $k$:th curve

goes from $E_k = (-n-1+k, 1/2-k)$ to $A_k = (n+1-k, 1/2-k)$ in the original coordinate system. We think of $h_k(x,n)$ as the height of the level-$k$ growth process at the point $x$ at time $t = n$. The level-$k$ height curve does not intersect the level-$(k+1)$ height curve. Note that the vertical steps always have step size $\pm 1$; $h_k(x+,n) - h_k(x-,n) \in \{-1,0,1\}$, and the jumps can occur only at points $x \in \{-n-1/2+k, -n+1/2+k, \dots, n+1/2-k\}$.

A random tiling of the Aztec diamond can be generated by the so called *shuffling algorithm* which we will now describe briefly, see [15] and [36] for more details. Start with $A_1$. We can tile it by either two horizontal dominoes, with probability $1-q$, or two vertical dominoes, with probability $q$, (compare (2.19). Assume that we have a random tiling of $A_k$ for some $k \geq 1$. We will define a random tiling of $A_{k+1}$, given a tiling of $A_k$. Call two dominoes which share a side of length two a *pair*. Two horizontal dominoes form a bad pair if the lower one is N and the upper one is S. Similarly, a pair of vertical dominoes is bad if the left one is E and the right one is W. All other pairs are *good*. In the first step we remove all bad pairs of dominoes. In the second step, for each domino in a good pair we move one unit step, upwards if it is N, downwards if it is S, to the left if it is W and to the right if it is E. After these two steps what remains to completely fill $A_{k+1}$ are $2 \times 2$-blocks. In the third and last step we fill each $2 \times 2$-block with a vertical pair with probability $q$ and with a horizontal pair with probability $1-q$. This generates a random tiling of the Aztec diamond $A_{k+1}$ with the probability (2.1), where $q = w^2(1+w^2)^{-1}$.

The shuffling algorithm translates into a PNG-type growth procedure for the cascade of height functions $\{h_k(x,n)\}_{1\leq k\leq n}$ defined above. We will now define the discrete PNG-type growth model which we obtain. We have shifted the picture $1/2$ unit upwards compared to the one we obtained from the modified DR-paths above. The "height paths" are built from *plus steps*, which go from $(m,n)$ to $(m+1,n+1)$ and consists of straight line segments between the points $(m,n)$, $(m+1/2,n)$, $(m+1/2,n+1)$ and $(m+1,n+1)$, *minus steps*, which go from $(m,n)$ to $(m+1,n-1)$ and consists of straight line segments between the points $(m,n)$, $(m+1/2,n)$, $(m+1/2,n-1)$ and $(m+1,n-1)$. Finally, we have *zero steps*, which are line segments from $(m,n)$ to $(m+2,n)$. The initial configuration, $t=0$, has zero steps between $(n,0)$ and $(n+2,0)$, $n \in 2\mathbb{Z}+1$. At time $t=k$, $k \geq 0$ we do the following:

(i) remove all zero steps;
(ii) move all plus steps one unit to the left and all minus steps one unit to the right;
(iii) if a plus step and a minus step pass each other in step (ii) they are removed;
(iv) add zero steps so that we obtain a connected curve from $(-\infty,0)$ to $(\infty,0)$;
(v) replace each zero step between $-(k+1)$ and $k+1$ with a combined plus and minus step independently with probability $q$.

We can define a *cascade of height curves* as follows. The level-$m$ curve initially has just zero steps between $(n,-(m-1))$ and $(n+2,-(m-1))$, $n \in 2\mathbb{Z}+1$. At each time step we apply the discrete PNG growth procedure independently to the levels $1,2,\dots$ with the condition that the level-$m$ curve cannot touch or intersect the level-$(m+1)$ curve, $m \geq 1$. If that happens in the random growth step, then this growth event is suppressed. Note har only a finite number of levels are changed at time $k$. The shuffling procedure induces an evolution of the modified DR-paths, and this is exactly the cascade of PNG growth models just defined. A plus step

corresponds to a W-domino, a minus step to an E-domino and a zero step to an S-domino. That the level-1 modified DR-paths evolves exactly according to the PNG-growth rules is immediate by comparison. All dominoes above the level-1 DR-path are N-dominoes and move upwards one step. Removing all bad vertical pairs corresponds to the annihilation (step (iii)) in the PNG-growth rule. Filling in $2 \times 2$-blocks is exactly the random growth, step (v) in the PNG growth rule. A somewhat more elaborate argument shows that the whole shuffling algorithm corresponds to the cascade defined above. We will not give the details.

Consider the level-1 DR-paths of type I, i.e. the upmost one. In CS-I it goes from $(1, 0)$ to $(n + 1, n)$ through the points $(i_k, j_k)$, $1 \le k \le p$, where $(i_0, j_0) = (1, 0)$, $(i_p.j_p) = (n+1, n)$ and $(i_{k+1}, j_{k+1}) - (i_k, j_k) = (1, 0)$, $(0, 1)$ or $(1, 1)$. The dominoes in the north polar zone are the dominoes above this path. Set

$$n - \lambda_\ell = \max\{j_k\, ;\, i_k = \ell\}, \quad 1 \le \ell \le n + 1. \tag{2.68}$$

Then $\lambda = (\lambda_1, \dots, \lambda_{n+1})$ is a partition and this is the partition associated with the north polar zone in [36]. (We can see this partition by marking each domino in the north polar zone with a point at the center. These points will lie in the integer lattice $\mathcal{L}_I$ in CS-I.) Set

$$\Lambda(n) = \{(i, j) \in \mathbb{Z}_+^2\, ;\, 1 \le j \le \lambda_i,\, 1 \le i \le n + 1\},$$

which is a random subset of $\mathbb{Z}_+^2$.

Let $w(i, j))_{(i,j)\in\mathbb{Z}_+^2}$ be independent geometric random variables with parameter $q$, $P[w(i, j) = k] = (1 - q)q^k$, $k \ge 0$, and define

$$G(M, N) = \max_{\pi} \sum_{(i,j)\in\pi} w(i, j), \tag{2.69}$$

where the maximum is over all up/right paths from $(1, 1)$ to $(M, N)$, see [37]. Set

$$\Omega(n) = \{(i, j) \in \mathbb{Z}_+^2\, ;\, G(M, N) + M + N - 1 \le n\}. \tag{2.70}$$

We get $\Omega(n + 1)$ from $\Omega(n)$ by independently adding a point with probability $p = 1 - q$ to every corner in $\Omega(n)$, see [37], and because of this we call it the *corner growth model*. It is proved in [36] that if the probability on $\mathcal{T}(A_n)$ is defined by (2.1) with $q = w^2(1 + w^2)^{-1}$, then the random sets $\Lambda(n)$ and $\Omega(n)$ have the same distribution.

Let $h(x, n)$ be the level-1 height function in the PNG growth model defined above. We can relate the distribution function for this height variable to the distribution function of $G(M, N)$. Let $P_0 = \frac{1}{2}(n + 2, n)$ be the midpoint in CS-I of the line segment from $(1, 0)$ to $(n + 1, n)$ and set $P_k = P_0 - k(1/2, 1/2)$, $k = 0, \pm 1, \dots, \pm n$. Assume that both $k$ and $n$ are even. Then $h(k, n) \le 2m - 1$ if and only if $P_k + m(-1, 1)$ is a point above the level-1 height curve, which happens if and only if

$$(\frac{n - k}{2} - m + 1, \frac{n + k}{2} - m + 1) \in \Omega(n).$$

Consequently, for $k$ and $n$ even,

$$P[h(k, n) \le 2m - 1] = P[G(\frac{n - k}{2} - m + 1, \frac{n + k}{2} - m + 1) \le 2m - 1]. \tag{2.71}$$

Using this relation and the asymptotic results for $G(M,N)$ in [37] we can show that the fluctuations of the height (and hence of the temperate zone since the height describes the boundary of the temperate zone) are of order $n^{1/3}$ and the appropriately rescaled fluctuations converges to the Tracy-Widom distribution (1.11).

As we have seen above the DR-paths can also be related to the zig-zag particle configurations. Using this we can relate the distribution function for $G(M,N)$ to the distribution of the rightmost particle in the Krawtchouk ensemble.

**Lemma 2.9.** *If $K = t + N + M - 1$, then*

$$P[G(M,N) \le t] = P_{Kr,M,K,q}[\max_{1\le j\le M} h_j \le t + M - 1].$$

*Proof.* Considerthe zig-zag path $Z_r$ as defined above. It maps to the zig-zag particle configuration $(h_1,\dots,h_r)$ with $h_1 < \dots < h_r$. The relation to the partition $\lambda$ defined by (2.68) is that $\lambda_r = n - h_r$. Hence, $G(r,x) + r + x - 1 \le n$, i.e. $\lambda_r \ge x$ in the tiling of $A_n$, if and only if $h_r \le n - x$. Thus, by theorem 2.2,

$$P[G(r,x) + r + x - 1 \le n] = P_{\mathrm{Kr},M,K,q}[\max_{1\le j\le M} h_j \le n - x],$$

and this translates into (2.9). □

We can now apply the edge scaling result for the Krawtchouk ensemble. Using the integral formula for the Krawtchouk polynomials (2.7) and proceeding in the same way as in [37] for the Meixner polynomials, we can prove that if $pt < q(1-t)$, $M = [Kt]$, $0 < t < 1$, then

$$\lim_{K\to\infty} P_{\mathrm{Kr},M,K,q}[\max_{1\le j\le M} h_j \le K\beta(t) + \xi\rho(t)K^{1/3}] = F(\xi) \tag{2.72}$$

for each $\xi \in \mathbb{R}$. Here $F(\xi)$ is given by (1.11) and

$$\beta(t) = (1-t)p + tq + 2\sqrt{pqt(1-t)},$$

$$\rho(t) = \left(\frac{pq}{t(1-t)}\right)^{1/6} (\sqrt{p(1-t)} + \sqrt{qt})^{2/3} (\sqrt{q(1-t)} - \sqrt{pt})^{2/3}.$$

We can now combine (2.9) and (2.72) (allowing a somewhat more complicated relation between $M, K$ and $t$) to give a new proof of theorem 1.2 in [37]. Note that in the derivation of (2.9) we have *not* used the RSK-correspondence which was central to the approach in [37].

Let $L(\alpha)$ denote the length of a longest increasing subsequence in a random permutation $\sigma$ from $S_N$ where $N$ is a Poisson random variable with mean $\alpha$. Then

$$P[L(\alpha) \le n] = \lim_{N\to\infty} P[G(N,N) \le n]$$

if we take $q = \alpha/N^2$, see [38]. Now, by (2.9),

$$\begin{aligned} P[L(\alpha) \le n] &= \lim_{N\to\infty} P_{\mathrm{Kr},N,n+2N-1,\alpha/N^2}[\max_{1\le j\le M} h_j \le n + N - 1] \\ &= \lim_{N\to\infty} \det(I - K_{\mathrm{Kr},N,n+2N-1,\alpha/N^2})_{\ell^2(\{n+N,\dots,n+2N-1\})}. \end{aligned} \tag{2.73}$$

Using the formulas (2.7) and (2.9), it follows that the last expression of (2.73) equals

$$\det(I - B_\alpha)_{\ell^2(\{n,n+1,\dots\})},$$

where $B_\alpha$ is the *discrete Bessel kernel*,

$$B_\alpha(x,y) = \sqrt{\alpha}\frac{J_x(2\sqrt{\alpha})J_{y+1}(2\sqrt{\alpha}) - J_{x+1}(2\sqrt{\alpha})J_y(2\sqrt{\alpha})}{x-y},$$

and we have rederived a result in [6] and [38]. Precise asymptotics for $L(\alpha)$ was first studied in [2]. We see that the longest increasing subsequence problem can be found in a limit of the Aztec diamond. In the same limit the discrete PNG model defined above, appropriately rescaled, converges to the PNG model studied in [57].

2.5. **Zig-zag paths for domino tilings of the plane.** Consider the squares $(m,n)+[-1/2,1/2]^2$, where $(m,n)\in\mathbb{Z}^2$. A domino tiling of the plane, which we identify with $\mathbb{C}$, is a covering of the whole plane by $2\times 1$ or $1\times 2$ rectangles, dominoes, where each domino covers exactly two of the basic squares. We can also think of this as a dimer configuration of the graph with vertices $(m,n)$ and edges between nearest neighbour vertices. A domino covers the neighbouring squares with centers $P$ and $Q$ if and only if the edge between $P$ and $Q$ are covered by a dimer. We will switch between the domino and dimer languages whenever it is convenient. Colour the points $(m,n)$ with $m+n$ even black and the other points white, and give the corresponding square the same colour. Consider the line $y=-x$. A domino tiling of the plane induces an infinite zig-zag path around black squares in complete analogy to the zig-zag paths in the Aztec diamond. We can map the zig-zag path to a particle configuration by saying that we have a particle at $x$ if and only if the zig-zag path goes from $x-1/2-i(x-1/2)$ to $x+1/2-i(x+1/2)$ via $x+1/2+i(-x+1/2)$, i.e. an east-south step. Note thst we have a particle at $x$ if and only if either the edge from $x-1-ix$ to $x-ix$ or the edge from $x-i(x+1)$ to $x-ix$ is covered by a dimer. In this way we get a particle configuration in $\mathbb{Z}$.

There is a unique translation invariant measure $\mu$ of maximal entropy, the *Burton-Pemantle measure*, on the space of domino tilings of the plane, see [9] and [45]. This measure induces a probability measure on zig-zag paths and hence on particle configurations in $\mathbb{Z}$; we get a point process on $\mathbb{Z}$. We want to show that this is a determinantal point process, [62], given by the discrete sine kernel. Let $E$ be a set of disjoint edges, i.e. they do not share a vertex, in the $\mathbb{Z}^2$-graph and let $U_E$ be the set of dimer configurations which contain $E$. Let $P$ be a white vertex and give the edge between $P$ and $P+z$ the *weight* $z$, where $z=\pm 1,\pm i$. Assume that the edges in $E$ cover the black vertices $b_1,\dots,b_k$ and the white vertices $w_1,\dots,w_k$. It is proved in [45],[46], using techniques by Kasteleyn, [44], that

$$\mu(U_E)=a_E\det(P(b_i-w_j))_{i,j=1}^k,$$

where $a_E$ is the product of the weights of the edges in $E$ and

$$P(x+iy)=\frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\frac{e^{i(x\theta-y\phi)}}{2i\sin\theta+2\sin\phi}d\theta d\phi.$$

Using this we can prove the following result.

**Theorem 2.10.** *The probability of having particles at positions $x_1,\dots,x_m$ in the zig-zag point process defined above is*

$$P[x]=\det\left(\frac{\sin\frac{\pi}{2}(x_j-x_k)}{\pi(x_j-x_k)}\right)_{j,k=1}^m. \tag{2.74}$$

*Proof.* If we have a particle at $x$, then one of the edges $x-ix, x-ix-1$ or $x-ix,x-ix-i$ is covered by a dimer. We take $E_z$ to be the set of edges $x_j-ix_j, x_j-ix_j-z_j$, where $z_j=1$ or $=i$, $1\le j\le m$. Then, $a_E=z_1\dots z_m$, $b_j=x_j-ix_j$ and $w_k=x_k-ix_k-z_k$, so that

$$P(b_j-w_k)=P(x_j-x_k-i(x_j-x_k)+z_k).$$

Thus,

$$P[x] = \sum_{z_j=1 \text{ or } i} \mu(U_{E_z}) = \sum_{z_j=1 \text{ or } i} z_1 \dots z_m \det(P(x_j - x_k - i(x_j - x_k) + z_k))$$
$$= \sum_{\sigma \in S_m} \text{sgn}\,(\sigma) \sum_{z_j=1 \text{ or } i} \prod_{j=1}^{m} z_{\sigma(j)} P(x_j - x_{\sigma(j)} - i(x_j - x_{\sigma(j)}) + z_{\sigma(j)})$$
$$= \sum_{\sigma \in S_m} \text{sgn}\,(\sigma) \prod_{j=1}^{m} K(x_j - x_{\sigma(j)}) = \det(K(x_j - x_k))_{j,k=1}^{m}, \tag{2.75}$$

where

$$K(u) = \sum_{z_j=1 \text{ or } i} zP(u - iu + z) = P(u + 1 - iu) + iP(u + (-u+1)i).$$

A computation shows that $P(-y - ix) = i(-1)^y P(x + iy)$ and thus

$$P(u + 1 - iu) = i(-1)^{-u-1} P(u - i(u+1)) = -i(-1)^u P(u - i(u+1)).$$

We obtain

$$\begin{aligned} K(u) &= i(P(u + i(-u+1)) - (-1)^u P(u - i(u+1)) \\ &= \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \frac{e^{i(u\theta + (u-1)\phi)} - (-1)^u e^{i(u\theta+(u+1)\phi)}}{2i \sin\theta + \sin\phi} d\theta d\phi \\ &= \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \frac{e^{iu(\theta+\phi)}(e^{-i\phi} - (-1)^u e^{i\phi})}{2i \sin\theta + \sin\phi} d\theta d\phi \\ &= \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} \frac{e^{i\theta u}(e^{-i\phi} - (-1)^u e^{i\phi})}{2i \sin(\theta - \phi) + 2\sin\phi} d\theta d\phi \end{aligned}$$

Set

$$G(\theta, u) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{-i\phi} - (-1)^u e^{i\phi}}{2i \sin(\theta - \phi) + 2\sin\phi} d\phi$$

so that

$$K(u) = \frac{i}{2\pi} \int_{-\pi}^{\pi} e^{i\theta u} G(\theta, u) d\theta.$$

If we write

$$G(\theta, u) = \frac{1}{2\pi i} \int_\gamma \frac{1 - (-1)^u z^2}{e^{i\theta} + i - (e^{-i\theta} + i)z^2} \frac{dz}{z}$$

we can use residue calculus to see that

$$G(\theta, u) = \begin{cases} \frac{(-1)^u}{e^{-i\theta}+i} & \text{if } -\pi < \theta < 0 \\ \frac{1}{e^{i\theta}+i} & \text{if } 0 < \theta < \pi \end{cases}. \tag{2.76}$$

Thus,

$$\begin{aligned} K(u) &= \frac{i}{2\pi} \int_{-\pi}^{0} \frac{(-1)^u e^{iu\theta}}{e^{-i\theta} + i} d\theta + \frac{i}{2\pi} \int_0^{\pi} \frac{e^{iu\theta}}{e^{i\theta} + i} \\ &= i \frac{1 - (-1)^u}{2\pi u} = \frac{\sin \frac{\pi u}{2}}{\pi u} (-1)^{u/2}. \end{aligned}$$

Inserting this in (2.75) gives

$$p[x] = \det\left(\frac{\sin\frac{\pi}{2}(x_j - x_k)}{\pi(x_j - x_k)}(-1)^{\frac{x_j - x_k}{2}}\right)_{j,k=1}^{m} = \det\left(\frac{\sin\frac{\pi}{2}(x_j - x_k)}{\pi(x_j - x_k)}\right)_{j,k=1}^{m}.$$

□

If we consider the zig-zag path through the center of the Aztec diamond, the measure on the zig-zag particle configurations has determinantal correlation functions, (2.8), by theorem 2.2. Take $r = n/2$ and $p = q = 1/2$. In this case the equilibrium measure, (2.26), has density $\rho'(x) = 1/2$, $0 \le x \le 1$. Hence, we can use lemma 2.8 to show that the limiting point pricess, as $n \to \infty$, has determinantal correlation functions with kernel

$$K(x,y) = \frac{\sin\frac{\pi}{2}(x-y)}{x-y},$$

i.e. exactly the same as in theorem 2.10. This is consistent with the conjecture, [36], [10] that in the center of the Aztec diamond a random tiling looks like a tiling of the plane under the Burton-Pemantle measure.

## 3. The Schur measure and non-intersecting paths

In section 2 we obtained the distribution function for the last-passage random variable $G(M,N)$, (2.69), using the non-intersecting paths in the Aztec diamond. It is natural to inquire whether the Meixner ensemble which is used to study $G(M,N)$ in [37] can also be obtained in a natural way using non-intersecting paths. The picture will again be a cascade of PNG-type growth models but different from the one studied in sect. 2.4.

We will define a certain random growth model, or rather a cascade of growth models, such that the probability distribution of the heights above the origin is given by the Schur measure introduced by Okounkov, [55]. The cascade of growth models is actually equivalent with the Robinson-Schensted-Knuth correspondence. Viennot, [Vi], gave a geometric construction of the RSK-correspondence for permutations, often called the "shadow" construction, see also [60], sect. 3.8. A permutation in $S_n$ can be described by putting $n$ points randomly in the unit square, Hammersley's picture, [31]. If one applies the first step in Viennot's construction and interprets the paths (shadow lines) as space-time paths one gets the polynuclear growth model (PNG) as introduced in [57] by Prähofer and Spohn. Thus we can equivalently think in terms of a growth model. If one takes the full Viennot construction one gets a cascade of growth models as proved by Okounkov, [56]. Okounkov did not base his presentation on Viennot's construction, but instead used the formulation of the RSK-correspondence in [8]. There is a generalization of Viennot's construction to the case of an integer matrix (generalized permutation) called the "matrix ball construction", see [26]. By the same argument, this can be translated into a growth model and will lead to the Schur measure. This growth model is also given in [56]. We will present a somewhat modified version of this growth model, which avoids the limiting procedure in [56]. The interesting thing is that this version leads directly to families of non-intersecting paths, namely the standard ones which can be used to obtain the Schur polynomials, [60], [64].

Let $W = (w(i,j))_{i,j=1}^{n}$ be an $n \times n$-matrix with non-negative integer elements, and set $w(i,j) = 0$ if $n \notin \{1,\dots,n\}^2$. The integer-valued *height functions* in the cascade of growth models are denoted by $h_k(x,t)$, $1 \le k \le n$, where $h_k(x,t)$ is

the height above $x \in \mathbb{R}$ at time $t \in \mathbb{N}$ of the *level $k$ growth process*. The height curves $x \to h_k(x,t)$ do not intersect, $h_k(x,t) - h_{k+1}(x,t) \geq 1$, $1 \leq k \leq n$ for all $x$ and $t$. The height curves will grow by the addition of unit squares and the growth procedure is defined as follows.

The vertical "sides" of $h_k(x,t)$ will be labelled. We can think of the curve $x \to h_k(x,t)$ as a lattice path starting at $(-k+1, -2n+1/2)$, ending at $(-k+1, 2n-1/2)$ and taking unit steps up, to the right or down. Each unit step up is labelled by $a_j$ for some $j$, $1 \leq j \leq n$ and each unit step down is labelled by $b_k$ for some $k$, $1 \leq k \leq n$. Call these vertical sides *left* and *right vertical sides* respectively. At time 0, $h_k(x,0) = -(k-1)$, $1 \leq k \leq n$ and there are no vertical sides.

Assume that $h_k(x,t)$ has been defined for $t \leq m-1$, $1 \leq k \leq n$ with labels on the vertical sides and such that the distance between a left vertical and a right vertical side is always odd. Furthermore $h_k(x,t) - h_{k+1}(x,t) \geq 1$. We will define $h_k(x,m), 1 \leq k \leq n$ so that it has the same properties. For each $x \in \mathbb{Z}$, $n \in \mathbb{N}$ and $1 \leq k \leq n$, we have a set $\mathcal{B}^{(k-1)}(x,t)$ of unit squares with labelled vertical sides. $\mathcal{B}^{(0)}(x,t)$ contains $w(i,j)$ unit squares with the left vertical side labelled $a_i$ and the right vertical side labelled $b_j$, where $i = (t+x+1)/2$, $j = (t-x+1)/2$. Recall that $w(i,j) = 0$ if $n \notin \{1, \dots, n\}^2$ so that $\mathcal{B}^{(0)}(x,t)$ is empty for odd $x$ at odd times $t$, and for even $x$ at even times $t$. The $\mathcal{B}^{(k-1)}(x,t)$, $k \geq 2$ are defined recursively in the growth procedure. Assume that $\mathcal{B}^{(\ell-1)}(x,m)$ has been defined for some $\ell$, $1 \leq \ell \leq n$. We will define $h_\ell(x,m)$ and $\mathcal{B}^{(\ell)}(x,m)$.

(**Horizontal growth**). Move each left vertical side in $h_\ell(x,m-1)$ one unit to the left, and each right vertical side one unit to the right, fill in with horizontal segments so that we get a connected curve, and denote the resulting height function by $h^*_\ell(x,m-1)$. The labels move together with the vertical sides. Since the distance between a left and a right vertical side is odd they cannot meet at the same point. Set $u = h_\ell(x-1,m-1) - h_\ell(x,m-1)$ and $v = h_\ell(x+1,m-1) - h_\ell(x,m-1)$. If $z = \min(u,v) > 0$, then a right vertical side, with labels $b_{r_1}, \dots, b_{r_u}$ (ordered in the upwards direction), will cross a left vertical side, with labels $a_{s_1}, \cdots_{s_v}$. If this happens we put $z$ unit squares in $\mathcal{B}^{(\ell)}(x,m)$ with vertical sides labelled $a_{r_j}, b_{s_j}$, $1 \leq j \leq z$. This defines $\mathcal{B}^{(\ell)}(x,m)$.

(**Vertical growth**). Next we put the labelled squares in $\mathcal{B}^{(\ell-1)}(x,m)$ on top of $h^*_\ell(x,m-1)$ at $x$ for all $x$, i.e. between $x-1/2$ and $x+1/2$. The result is $h_\ell(x,m)$. Note that all vertical sides are labelled, and that by the way that the $\mathcal{B}^{(\ell-1)}(x,m)$:s were defined, the distance between left and right vertical sides is always odd.

In this way we recursively define $h_k(x,t)$, with labelled vertical sides for $1 \leq k, t \leq n$, $x \in \mathbb{R}$.

Next we want to describe the final configuration. Let $\lambda = (\lambda_1, \dots, \lambda_n)$ be a partition and let $\mathcal{P}(\lambda; a)$ denote the set of all non-intersecting, labelled, up/right lattice paths $\Gamma = \{\Gamma_k\}_{k=1}^n$, where $\Gamma_k$ starts at $(-k+1, -2n+1/2)$ and ends at $(-1/2, \lambda_j - j + 1)$ and where all up-steps have an $x$-coordinate of the form $2(j-n) - 1/2$ for some $j$, $1 \leq j \leq n$. Each unit length of the vertical sides with the $x$-coordinate $2(j-n) - 1/2$ is labelled by $a_j$, $\leq j \leq n$. Let $\tilde{\mathcal{P}}(\lambda; b)$ be the paths we obtain by reflecting the paths in $\mathcal{P}(\lambda; a)$ in the $y$-axis and putting the label $b_k$ on each unit vertical side with $x$-coordinate $2(n-k) + 1/2$, $1 \leq k \leq n$. If we join a family of non-intersecting paths from $\mathcal{P}(\lambda; a)$ to a family from $\tilde{\mathcal{P}}(\lambda; b)$ by adding horizontal segments from $(-1/2, \lambda_j - j + 1)$ to $(1/2, \lambda_j - j + 1)$, $1 \leq j \leq n$, we

obtain a family of non-intersecting, labelled height curves. Let $\mathcal{H}(\lambda; a, b)$ denote all the families of non-intersecting height curves obtained in this way.

We claim that $H = \{h_k(x, 2n-1)\}_{k=1}^n$ constructed as above belongs to $\mathcal{H}(\lambda; a, b)$ if we put $\lambda_j = h_j(0, 2n-1) + j - 1$. We write $H = (\Gamma, \tilde{\Gamma})$, where $\Gamma \in \mathcal{P}(\lambda; a)$ and $\tilde{\Gamma} \in \tilde{\mathcal{P}}(\lambda; b)$. To see this note that a square with labels $a_j, b_k$ is introduced at level 1 at position $j-k$ at time $j+k-1$. In the remaining time $2n-1-(j+k-1) = 2n-j-k$, the left vertical side moves $2n-j-k$ steps to the left and the right vertical side $2n-j-k$ steps to the right. Thus the $a_j$-label ends up at a position with $x$-coordinate $j-k-1/2-(2n-j-k) = 2(j-n)-1/2$, and the $b_k$-label ends up at a position with $x$-coordinate $j-k+1/2+2n-j-k = 2(n-k)+1/2$. Thus all left (right) vertical sides end up to the left(right) of the origin at the correct positions. Thus, we obtain a map from the set of $n \times n$ integer matrices to $\cup_\lambda \mathcal{H}(\lambda; a, b)$, where the union is over all partitions $\lambda = (\lambda_1, \dots, \lambda_n)$. This map is invertible, and hence we obtain a bijection. That the map is invertible follows from the fact that the growth procedure can be reversed! We start at the bottom level, $h_n(x, t)$, but now right vertical sides move to the left and left vertical sides to the right. If they cross we move squares up to the next level, just as we moved them down before. At the upper level they tell us where we should split and introduce new left and right vertical sides. Those squares that are taken out at the top level give us the entries in the matrix. If we take out $m$ squares with labels $a_j, b_k$ we put the number $m$ at position $(j, k)$ in the matrix. (We can also find the position from the time $t$ when and the place $x$ where the squares are removed, $x = j-k$, $t = j+k-1$.) We will not describe all the details of this reverse procedure. In a sense the cascade of growth models records the history of the growth process. When two vertical segments pass each other at a certain level information is lost at this level, but this information is recorded at the next level.

If the $w(j,k)$:s are independent geometric random variables with $P[w(j,k) = m] = (1-a_jb_k)(a_jb_k)^m$, $m \geq 0$, then the probability of a particular integer matix $W = (w(j,k))_{j,l=1}^n$ is

$$\prod_{j,k=1}^{n} (1-a_jb_k) \prod_{j,k=1}^{n} (a_jb_k)^{w(j,k)} = \prod_{j,k=1}^{n} (1-a_jb_k)\omega(W),$$

where

$$\omega(W) = \prod_{j=1}^{n} a_j^{\sum_k w(j,k)} \prod_{k=1}^{n} b_k^{\sum_j w(j,k)}.$$

If we interpret the labels on the vertical sides of $\mathcal{P}(\lambda; a)$ as weights and define the weight, $\omega(\Gamma)$, of an element $\Gamma \in \mathcal{P}(\lambda; a)$ as the product of the weights of all vertical sides (horizontal sides have weight 1), then we see that the growth procedure defined above transports the weights in the correct way; if $W$ has weight $\omega(W)$, and $W$ maps to $(\Gamma, \tilde{\Gamma})$ in $\mathcal{H}(\lambda; a, b)$, then

$$\omega(W) = \omega(\Gamma)\omega(\tilde{\Gamma}).$$

The total weight of all up/right paths from $(-k+1, -2n+1/2)$ to $(-1/2, \lambda_j - j + 1)$ where all vertical sides have $x$-coordinates of the form $2(i-n) - 1/2$, $1 \leq i \leq n$, is

given by

$$\sum_{m_1+\cdots+m_n=\lambda_j-j+k} a_1^{m_1}\dots a_n^{m_n} = h_{\lambda_j-j+k}(a_1,\dots,a_n);$$

We have $m_i$ vertical steps with $x$-coordinate $2(i-n)-1/2$ and these have weight $a_i$. Here $h_m(a_1,\dots,a_n)$ is the complete symmetric polynomial of degree $m$ in $n$ variables; $h_m(a)\equiv 0$ if $m<0$.

The Lindström-Gessel-Viennot method, which is the discrete analogue of the Karlin-McGregor theorem, see for example [65], p. 98 for a precise statement, gives

$$\sum_{\Gamma\in\mathcal{P}(\lambda;a)} \omega(\Gamma) = \det(h_{\lambda_j-j+k}(a))_{j,k=1}^N = s_\lambda(a). \tag{3.1}$$

The expresion in the middle can be taken as the definition of the *Schur polynomial* $s_\lambda(a)$. The growth procedure above defines a map $S(W)=\lambda$ from the integer matrix $W$ to the partition $\lambda$ defined by the succesive heights. We obtain

$$\begin{aligned} P[S(W)=\lambda] &= \sum_{W:S(W)=\lambda}\prod_{j,k=1}^n(1-a_jb_k)\omega(W) \\ &= \prod_{j,k=1}^n(1-a_jb_k)\sum_{\Gamma\in\mathcal{P}(\lambda;a),\tilde\Gamma\in\tilde{\mathcal{P}}(\lambda;b)}\omega(\Gamma)\omega(\tilde\Gamma) \\ &= \left[\prod_{j,k=1}^n(1-a_jb_k)\right]s_\lambda(a)s_\lambda(b)\doteq P_{\text{Schur}}[\lambda], \end{aligned} \tag{3.2}$$

the *Schur measure* on partitions, introduced in [55].

**Remark 3.1.** Note that an element $\Gamma=\{\Gamma_k\}_{k=1}^n$ in $\mathcal{P}(\lambda;a)$ corresponds to a unique semi-standard tableaux $T$ with shape $\lambda$, $\operatorname{sh}(T)=\lambda$. If $\Gamma_k$ has $r_j$ vertical steps at $2(j-n)-1/2$, we put $r_j$ $j$:s, $j=1,\dots,n$ in weakly increasing order in the $\lambda_k$ boxes in row $k$. Similarly an element in $\tilde{\mathcal{P}}(\lambda;b)$ gives a semistandard tableaux $D$. Thus we obtain a one-to-one map $W\to(S,T)$, which is exactly the RSK-correspondence. If we let $m_j(T)$ denote the number of $j$:s in $T$, we see that we also have, by (3.1),

$$s_\lambda(a) = \sum_{\Gamma\in\mathcal{P}(\lambda;a)}\omega(\Gamma) = \sum_{T:\operatorname{sh}=\lambda} a_1^{m_1(T)}\dots a_n^{m_n(T)},$$

which is the combinatorial definition of the Schur polynomial, see [64].

Consider now the random variable $G(M,N)$ defined by (2.69). From its definition it is clear that we can compute $G(M,N)$ recursively by

$$G(M,N) = \max(G(M-1,N),G(M,N-1)) + w(M,N). \tag{3.3}$$

We want to show that, for $1\le M,N\le n$,

$$G(M,N) = h_1(M-N,M+N-1), \tag{3.4}$$

in particular $G(n,n)=h_1(0,2n-1)=\lambda_1$, which is a well known property of the RSK-correspondence, [60], [48]. It follows from the definition of the growth process

above that

(3.5)
$$h(x,t)=\max(h_1(x-1,t-1),h(x,t-1),h(x+1,t-1))+w(\frac{t+x+1}{2},\frac{t-x+1}{2}).$$

(Recall that $w(j,k)=0$ if $(j,k)\notin\{1,\dots,n\}^2$.) Write $\tilde{G}(j,k)=h(j-k,j+k-1)$. Then (3.5) becomes,

$$\tilde{G}(j,k)=\max(\tilde{G}(j-1,k),\tilde{G}(j-\frac{1}{2},k-\frac{1}{2}),\tilde{G}(j,k-1))+w(j,k). \tag{3.6}$$

We will now use induction. If $M+N-1=1$, then $M=N=1$ and $G(1,1)=w(1,1)=\tilde{G}(1,1)$. Assume that $G(M,N)=\tilde{G}(M,N)$ for $M+N-1<k$. We want to show that it is true for $M+N-1=k$. Note that, by (3.6),

(3.7)
$$\tilde{G}(M-\frac{1}{2},N-\frac{1}{2})=\max(\tilde{G}(M-\frac{3}{2},N-\frac{1}{2}),\tilde{G}(M-1,N-1),\tilde{G}(M-\frac{1}{2},N-\frac{3}{2}))$$

since $w(M-1/2,N-1/2)=0$. By our assumption and (3.6),

$$G(M-1,N)=\tilde{G}(M-1,N)\geq\tilde{G}(M-\frac{3}{2},N-\frac{1}{2}) \tag{3.8}$$

and

$$G(M,N-1)=\tilde{G}(M,N-1)\geq\tilde{G}(M-\frac{1}{2},N-\frac{3}{2}). \tag{3.9}$$

Furthermore, by our assumption, $\tilde{G}(M-1,N-1)=G(M-1,N-1)$ and $G(M-1,N)\geq G(M-1,N-1)$ by (3.3). Combining this with (3.7), (3.8) and (3.9) we see that

$$\max(G(M-1,N),G(M,N-1))\geq\tilde{G}(M-\frac{1}{2},N-\frac{1}{2}). \tag{3.10}$$

Consequently, by (3.6), our assumption and (3.10)

(3.11)
$$\begin{aligned}\tilde{G}(M,N)&=\max(G(M-1,N),\tilde{G}(M-\frac{1}{2},N-\frac{1}{2}),G(M,N-1))+w(M,N)\\&=\max(G(M-1,N),G(M,N-1))+w(M,N)=G(M,N),\end{aligned}$$

which completes the proof.

From (3.4) and (3.2) we obtain

$$P[G(M,N)\leq t]=[\prod_{j,k=1}^{n}(1-a_jb_k)]\sum_{\lambda:\lambda_1\leq t}s_\lambda(a)s_\lambda(b).$$

Using the fact that the Schur measure has determinantal correlation functions, which was proved in [55], see also [39], we see that this equals a Fredholm determinant with a certain kernel, and this can be exploited for the asymptotic analysis.

**Remark 3.2.** The case when $w(i,j)$ are independent exponential random variables with mean 1 can be obtained as a limit of the geometric case as discussed in [37]. We can take the same limit in the construction above and this leads to a continuous analogue of the RSK-correspondence. The resulting picture of paths can be viewed as two families of $n$ non-intersecting Poisson processes with rate $1/2$. (We take $a_j=1-1/2L$, $b_j=1-1/2L$, so that $a_jb_k\approx 1-1/L$ when $L$ is large. This gives the rate $1/2$ for the limiting Poisson processes on both sides.) Take the vertical

axis in the negative direction as time axis, and the horizontal axis as counting the number of events in the Poisson process. If the heights above the origin in the cascade are $t_1 > \cdots > t_n$, then the $k$:th process $A_k$ on the left (and right) start at 1 at the time $-t_k$ and end at $n+1-k$ at time 0. The probability for $X_i$ to go from 0 to $n-j$ in a time interval of length $t_i$ is

$$P[X_i(t_i) = n - j] = e^{-t_i/2}\frac{(t_i/2)^{n-j}}{(n-j)!}. \tag{3.12}$$

The Karlin/McGregor theorem can be generalized to unequal starting times, see [42], and we find that the probability for the $n$ non-intersecting paths to the left with the specified initial and final positions is

$$\det\left(e^{-t_i/2}\frac{(t_i/2)^{n-j}}{(n-j)!}\right)_{i,j=1}^{n} = 2^{-n(n+1)/2}\prod_{j=1}^{n}\frac{1}{j!}\Delta_n(t)\prod_{j=1}^{n}e^{-t_i/2}. \tag{3.13}$$

The possible heights lie in $[0,\infty)$, so we obtain the probability density

$$\frac{1}{Z_n}\Delta_n(t)^2\prod_{j=1}^{n}e^{-t_i}, \tag{3.14}$$

where

$$Z_n = \int_{[0,\infty)^n}\Delta_n(t)^2\prod_{j=1}^{n}e^{-t_i}d^nt.$$

Hence we have rederived the result of proposition 1.4 in [37] (in the case $M = N$) using non-intersecting paths. The probability density (3.14) is a special case of the Laguerre ensemble. It is also possible to consider the case when $w(i,j)$ is exponential with parameter $\alpha_i + \beta_j$. This leads to a continuous analogue of the Schur measure, which can also be obtained as a limit of the Schur measure defined above with $a_i = 1 - \alpha_i/L$, $b_i = 1 - \beta_i/L$ as $L \to \infty$. Using the methods of [39] we can derive the correlation functions for the continuous Schur measure, and we can also obtain proposition 1.4 in [37] in the case $M \neq N$.

## 4. Random walks and rhombus tilings of a hexagon

4.1. **Derivation of the Hahn ensemble.** In this section we will explore the relation between non-intersecting random walk paths and another tiling problem. We will consider random tilings of an abc-hexagon with rhombi, see [12], which are directly related to so called boxed plane partitions, [64]. An *abc-hexagon* has sides $a, b, c, a, b, c$ (in clockwise order) and equal angles. We want to tile this region with unit rhombi (often called lozenges) with angles $\pi/3$ and $2\pi/3$. The number of possible tilings is given by MacMahon's formula, [64],

$$N(a,b,c) = \prod_{i=1}^{a}\prod_{i=1}^{b}\prod_{i=1}^{c}\frac{i+j+k-1}{i+j+k-2}. \tag{4.1}$$

We obtain a random tiling by picking each tiling with equal probability. It is well known that a rhombus tiling can be described by non-intersecting random walk paths, see for example [18] and below. This is the approach we will adopt here. We will give a new derivation of the Hahn ensemble introduced in [38].

Let $a, b, c \geq 1$ be given integers. Take $\mathbf{e} = (0, 1/2)$ and $\mathbf{f} = (\sqrt{3}/2, 0)$ as basis vectors in our coordinate system; all coordinates will refer to this choice of basis

vectors. Our hexagon has corners at $P_1 = (0,0)$, $P_2 = (b,-b)$, $P_3 = (a+b, a-b)$, $P_4 = (a+b, a-b+2c)$, $P_5 = (a, a+2c)$ and $P_6 = (0, 2c)$. Consider random walks $S^k(m)$, $0 \le m \le a+b$, starting at $(0, 2(k-1))$ and ending at $(a+b, a-b+2(k-1))$, $1 \le k \le c$,

$$S^k(m) = (0, 2k) + \sum_{j=1}^{m} (1, X_j^k),$$

where $X_j^k = \pm 1$ are independent Bernoulli random variables taking each value with probability $1/2$. Assume that these random walks are non-intersecting. We restrict to the case $a \ge b$; the case $a \le b$ is completely analogous. Set

$$\alpha_m = \begin{cases} -m & \text{if } 0 \le m \le b \\ m - 2b & \text{if } b \le m \le a \\ m - 2b & \text{if } a \le m \le a+b, \end{cases}$$

$$\beta_m = \begin{cases} m + 2(c-1) & \text{if } 0 \le m \le b \\ m + 2(c-1) & \text{if } b \le m \le a \\ 2a - m + 2(c-1) & \text{if } a \le m \le a+b, \end{cases}$$

and $\gamma_m = (\beta_m - \alpha_m)/2$. Then $\alpha_m \le S^k(m) \le \beta_m$. Set $x_k = (S^k(m) - \alpha_m)/2$ and note that $0 \le x_k \le \gamma_m$, $1 \le k \le c$. These numbers describe the points where the random walks intersect a vertical line. Note the analogy with the previous problems. Think of $x_1, \dots, x_c$ as the positions of $c$ *particles* in a discrete gas confined to $\{0, \dots, \gamma_m\}$. Let $\xi_1 < \dots < \xi_{L_m}$,

$$L_m = \gamma_m + 1 - c = \begin{cases} m & \text{if } 0 \le m \le b \\ b & \text{if } b \le m \le a \\ a + b - m & \text{if } a \le m \le a+b, \end{cases} \tag{4.2}$$

be the positions of the *holes*. The holes correspond to the positions $(m, \alpha_m + 2\xi_k)$ in our coordinate system.

There are three types of rhombi. *Type I* which are spanned by $\mathbf{e} + \mathbf{f}$ and $2\mathbf{f}$, *Type II* which are spanned by $\mathbf{e} - \mathbf{f}$ and $2\mathbf{f}$ and *Type III* (called *vertical*) spanned by $-\mathbf{e} + \mathbf{f}$ and $\mathbf{e} + \mathbf{f}$. (We will sometimes call type I and II *horizontal*.) Given the non-intersecting random walk paths we can now tile the hexagon with rhombi as follows. If $X_{m+1}^k = 1$ we put a type I rhombus at $(m, S^k(m))$, and if $X_{m+1} = -1$ we put a type II rhombus at $(m, S^k(m))$. Finally we put a type III rhombus at each of the hole positions $(m, \alpha_m + 2\xi_k)$. Note that the vertical rhombi are associated with holes in the gas, whereas each particle is associated with a horizontal rhombus. Our non-intersecting random walks correspond to picking all of the possible hexagon tilings of the abc-hexagon with equal probability. We want to compute the probability measure induced on the particle/hole configurations on the vertical line $x = m$. In order to be able to formulate the results we first define the Hahn and associated Hahn ensembles.

The *Hahn ensemble*, [38] is a probability measure on $\{0, \dots, N\}^m$ defined by

$$P_{N,m}^{(\alpha,\beta)}[h] = \frac{1}{Z_{N,m}^{(\alpha,\beta)}} \Delta_m(h)^2 \prod_{j=1}^{m} w_N^{(\alpha,\beta)}(h_j), \tag{4.3}$$

where $\alpha, \beta > -1$ are given parameters,

$$w_N^{(\alpha,\beta)}(t) = \frac{(N+\alpha-t)!(\beta+t)!}{t!(N-t)!} \tag{4.4}$$

a weight function and

$$Z_{N,m}^{(\alpha,\beta)} = \sum_{h\in\{0,\dots,N\}^m} \Delta_m(h)^2 \prod_{j=1}^m w_N^{(\alpha,\beta)}(h_j) \tag{4.5}$$

a normalization constant. This ensemble is related to the Hahn polynomials, [54], which are orthogonal on $\{0,\dots,N\}$ with respect to the weight (4.4). For some facts about these polynomials see the proof of lemma 4.2 below. Using the leading coefficients of the normalized Hahn polynomials a standard computation, [52], gives

$$Z_{N,m}^{(\alpha,\beta)} = m! \prod_{j=0}^{m-1} \frac{j!(\alpha+j)!(\beta+j)!(\alpha+\beta+j+N+1)!(\alpha+\beta+j)!}{(\alpha+\beta+2j)!(\alpha+\beta+2j+1)!(N-j)!}. \tag{4.6}$$

The *associated Hahn ensemble* on $\{0,\dots,N\}^m$ is defined by

$$\tilde{P}_{N,m}^{(\alpha,\beta)}[h] = \frac{1}{\tilde{Z}_{N,m}^{(\alpha,\beta)}} \Delta_m(h)^2 \prod_{j=1}^m \tilde{w}_N^{(\alpha,\beta)}(h_j), \tag{4.7}$$

where $\tilde{Z}_{N,m}^{(\alpha,\beta)}$ is a normalization constant and

$$\tilde{w}_N^{(\alpha,\beta)}(t) = \frac{1}{t!(N-t)!(N+\alpha-t)!(\beta+t)!}. \tag{4.8}$$

The ensembles (4.3) and (4.7) are related by a particle/hole transformation, compare (2.17) above, and see the proof of theorem 4.1 below.

Let $\tilde{P}_m(x_1,\dots,x_c)$ denote the probability of having the particle configuration $x_1,\dots,x_m$ at time $m$ (along the vertical axis $x=m$), and $P(\xi_1,\dots,\xi_{L_m})$, $L_m$ given by (4.2), the probability of having the hole configuration $\xi_1,\dots,\xi_{L_m}$ at time $m$.

**Theorem 4.1.** *If $a, b, c \geq 1$ are given integers, $a \geq b$, and we define $a_m = |a-m|$, $b_m = |b-m|$, then*

$$\tilde{P}_m(x_1,\dots,x_c) = \tilde{P}_{\gamma_m,c}^{(a_m,b_m)}[x] \tag{4.9}$$

*and*

$$P_m(\xi_1,\dots,\xi_{L_m}) = P_{\gamma_m,L_m}^{(a_m,b_m)}[\xi] \tag{4.10}$$

*Proof.* The number of random walk paths from $j$ to $k$ in $m$ steps is ($m+k-j$ even),

$$\binom{m}{\frac{m+k-j}{2}} = e_{\frac{m+k-j}{2}}(1^m),$$

where $1^m = (1,\dots,1) \in \mathbb{N}^m$ and $e_n(x)$ is the elementary symmetric function. We can now use the Karlin-McGregor, Lindström-Gessel-Viennot argument to see that the number of non-intersecting random walk paths from $(0, 2(k-1))$ to $(m, 2x_k + \alpha_m)$ is

$$A_m(x) \doteq \det(e_{\delta_m+x_k-j+1}(1^m))_{j,k=1}^c, \tag{4.11}$$

where $\delta_m = (m + \alpha_m)/2$. Introduce the shifted particle positions

$$s_k = x_{c+1-k} + \delta_m \in \{0, \dots, \gamma_m + \delta_m\},$$

which means that we have introduced $\delta_m$ extra holes at the positions $j-1$, $1 \le j \le \delta_m$. Define the partition $\lambda$ by

$$\lambda_k = s_k + k - c, \quad 1 \le k \le c.$$

By reversing the order of the rows and columns in (4.11) we obtain

$$\begin{aligned} A_m(x) &= \det(e_{\delta_m + x_{c+1-k} - c + j}(1^m))_{j,k=1}^c \\ &= \det(e_{\lambda_k - k + j}(1^m))_{j,k=1}^c = s_{\lambda'}(1^m), \end{aligned} \tag{4.12}$$

where $\lambda'$ is the conjugate partition to $\lambda$ and $s_\lambda$ is the Schur polynomial, see (3.4) above and [60].

If we set $r_j = c + j - 1 - \lambda'_j$, $1 \le j \le L_m + \delta_m$, then $\{r_1, \dots, r_{L_m+\delta_m}\} \cup \{s_1, \dots, s_c\} = \{0, \dots, \gamma_m + \delta_m\}$ so the $r_k$ give the positions of the (shifted) holes including the extra holes. The positions of the original holes are given by

$$\xi_j = r_{\delta_m + j} - \delta_m, \quad 1 \le j \le L_m. \tag{4.13}$$

Note that $L_m + \delta_m = m$ if $0 \le m \le a$ and $L_m + \delta_m = a$ if $a \le m \le a+b$. Let $\mu = \lambda' = (\lambda'_1, \dots \lambda'_m)$ if $0 \le m \le a$, and $\mu = (\lambda'_1, \dots \lambda'_m, 0, \dots, 0)$ ($m - a$ extra zeros) if $a < m \le a + b$. Then,

$$A_m(x) = s_{\lambda'}(1^m) = s_\mu(1^m) = \prod_{1 \le i \le j \le m} \left( \frac{\mu_i - \mu_j + j - i}{j - i} \right), \tag{4.14}$$

by the classical formula for a Schur polynomial. We now want to rewrite the right hand side of (4.14) in terms of $\xi$ using (2.17) and (4.13). Some computation gives

$$A_m(x) = C(m, a, b, c)\Delta_{L_m}(\xi), \tag{4.15}$$

if $0 \le m \le b$,

$$A_m(x) = C(m, a, b, c) \prod_{j=1}^{L_m} \frac{(\xi_j + m - b)!}{\xi_j!} \Delta_{L_m}(\xi), \tag{4.16}$$

if $b \le m \le a$, and

$$A_m(x) = C(m, a, b, c) \prod_{j=1}^{L_m} \frac{(\xi_j + m - b)!(b + c - 1 - \xi_j)}{\xi_j!(a + b + c - m - 1 - \xi_j)!} \Delta_{L_m}(\xi), \tag{4.17}$$

if $a \le m \le a + b$, where $C(m, a, b, c)$ is a constant, e.g.

$$C(m, a, b, c) = \prod_{j=0}^{L_m - 1} \frac{1}{(j + m - b)!} \tag{4.18}$$

for $b \le m \le a$.

We will also write $W_m(\xi) = A_m(x) =$ the number of non-intersecting random walks ending with hole configuration $\xi$ at time $m$. The number of possible non-intersecting random walks coming from the right side and going in the other direction is $W_{m'}(\xi')$, where $\xi'_j = c + L_m - 1 - \xi_{L_m+1-j}$ and $m' = a + b - m$. This follows from the symmetry of the hexagon. The total number of tilings, given $\xi$, at time

$m$ is then $W_m(\xi)W_{m'}(\xi')$. This can be computed by using (4.15) - (4.17) with the result

$$W_m(\xi)W_{m'}(\xi') = C^*(m,a,b,c)\Delta_{L_m}(\xi)^2 \prod_{j=1}^{L_m} w_{\gamma_m}^{(a_m,b_m)}(\xi_j), \tag{4.19}$$

where $C^*(m,a,b,c)$ is a constant. Using (4.18) we see that, for $b \le m \le a$

$$C^*(m,a,b,c) = C(m,a,b,c)C(m',a,b,c) = \prod_{j=0}^{b-1} \frac{1}{(j+m-b)!(j+a-m)!}. \tag{4.20}$$

It follows that the total number of tilings is

$$\begin{aligned} N(a,b,c) &= C^*(m,a,b,c) \sum_{\xi \in \{0,\dots,\gamma_m\}^{L_m}} \Delta_{L_m}(\xi)^2 \prod_{j=1}^{L_m} w_{\gamma_m}^{(a_m,b_m)}(\xi_j) \\ &= C^*(m,a,b,c) Z_{\gamma_m,L_m}^{(a_m,b_m)}, \end{aligned}$$

by (4.5). This proves (4.10). Note that by combining (4.20) and (4.6) we obtain, after some computation (where we take $m = b$),

$$N(a,b,c) = \prod_{j=0}^{b-1} \frac{j!(a+c+j)!}{(a+j)!(c+j)!},$$

which proves MacMahon's formula (4.1).

We now want to go from the variables $\xi$ to $x$. Since $\{x_1,\dots,x_c\} \cup \{\xi_1,\dots,\xi_{L_m}\} = \{0,\dots,\gamma_m\}$ we can use (2.17) to get

$$\Delta_{L_m}(\xi) = \left(\prod_{j=1}^{\gamma_m} j!\right) \left(\prod_{k=1}^{c} \frac{1}{x_k!(\gamma_m - x_k)!}\right) \Delta_c(x).$$

Using this it is straightforward to show that

$$W_m(\xi)W_{m'}(\xi') = C_*(m,a,b,c)\Delta_c(x)^2 \prod_{j=1}^{c} \tilde{w}_{\gamma_m}^{(a_m,b_m)}(x_j),$$

and (4.9) follows, □

4.2. **Some asymptotic results.** Random tilings of a hexagon shows the same type of arctic ellipse effect as the Aztec diamond, see .... We will have polar zones associated with each corner of the hexagon. Consider the vertex $P_6 = (0, 2c)$ of the hexagon. We will say that two type I rhombi are *adjacent* if they share an edge. The *polar zone* associated with $P_6$ is now defined as follows. If there is no type I rhombus $R_0$ having $P_6$ as a vertex, the polar zone is empty. Otherwise a type I rhombus $R$ belongs to the polar zone if there is a sequence of rhombi $R_0, \dots, R_k = R$ such that $R_j$ and $R_{j+1}$ are adjacent. Consider the horizontal rhombi immediately to the left of the line $x = m$, the *m:th column*, there are $c$ of them. If $Z_m = \xi_{L_m} = \max \xi_j$ is the position of the last hole, i.e. vertical rhombus on the line $x = m$, then all the horizontal rhombi above it in the $m$:th column are of type I and belong to the polar zone of $P_6$. Hence, the boundary of this polar zone is obtained by joining the points $A_1, B_1, \dots, A_a, B_a$, where $A_m = (m-1, \alpha_m + 2Z_m + 1)$, $B_m = (m, \alpha_m + 2Z_m)$ with straight line segments. The boundary of this polar zone is thus related to the position of the rightmost particle in the Hahn ensemble.

The asymptotic position of the rightmost particle and its large deviation properties can be investigated using the results of [37], sect. 2.2. Consider the Hahn ensemble (4.3). Assume that $m/N \to t \in (0,1)$ and $\frac{1}{N}(\alpha,\beta) \to (\alpha_0,\beta_0)$ as $N \to \infty$. We have the limit

$$V(s) = -\lim_{N\to\infty} \frac{1}{m} W_N^{(\alpha,\beta)}(ms) = -\frac{1}{t}U(ts),$$

where

$$\begin{aligned} U(s) &= (1+\alpha_0-s)\log(1+\alpha_0-s) + (\beta_0+s)\log(\beta_0+s) \\ &\quad - s\log s - (1-s)\log(1-s) - C. \end{aligned}$$

Here we have introduced the modified weight,

$$W_N^{(\alpha,\beta)}(x) = \frac{\binom{\beta+x}{x}\binom{\alpha+N-x}{x}}{\binom{\alpha+\beta+N+1}{N}}, \tag{4.21}$$

which only differs by a multiplicative constant. The *equilibrium measure*, $u_{\mathrm{eq}}^{(t,\alpha_0,\beta_0)}(s)ds$, for the Hahn ensemble is the unique solution of the constrained, weighted variational problem

$$F_V = \inf_{u\in\mathcal{A}} \left( \int_0^{1/t}\int_0^{1/t} \log|\sigma-s|^{-1} u(\sigma)u(s)d\sigma ds + \int_0^{1/t} V(s)u(s)ds \right), \tag{4.22}$$

where $\mathcal{A} = \{u \in L^1[0,1/t]\,;\, \int_0^{1/t} u = 1 \text{ and } 0 \le u \le 1\}$. From [37], theorem 2.2, we obtain the following large deviation result. Let $R = R(t,\alpha_0,\beta_0)$ be the right endpoint of the support of the equilibrium measure, and let $\epsilon > 0$ be given. There are functions $L(R-\epsilon)$ and $J(R+\epsilon)$ such that

$$\lim_{N\to\infty} \frac{1}{N^2} \log P_{N,m}^{(\alpha,\beta)}[\frac{1}{m}\max h_j \le R-\epsilon] = -2L(R-\epsilon), \tag{4.23}$$

and, if $J(R+\epsilon) > 0$ for $\epsilon > 0$, then

$$\lim_{N\to\infty} \frac{1}{N} \log P_{N,m}^{(\alpha,\beta)}[\frac{1}{m}\max h_j \ge R+\epsilon] = -2J(R+\epsilon). \tag{4.24}$$

We always have $L(R-\epsilon) > 0$ if $\epsilon > 0$, but we must prove that $J(R+\epsilon) > 0$ if $\epsilon > 0$. The function $J$ is defined by

$$J(x) = \inf_{\tau \ge x} g(x),$$

where

$$g(x) = \int_0^\infty \log|x-y|^{-1} u_{\mathrm{eq}}(y)dy + \frac{1}{2}V(x) + \frac{1}{2}\int_0^\infty V(y)u_{\mathrm{eq}}(y)dy - F_V. \tag{4.25}$$

By the general theory for the constrained variational problem (4.22), see [13], $g(x) \ge 0$ for $x \ge R$. Now, for $x > R$,

$$g''(x) = \int_0^\infty \frac{u_{\mathrm{eq}}(y)}{(x-y)^2}dy + \frac{1}{2}V''(x)$$

and $V''(x) = -tU''(tx)$ with

$$U''(s) = -\frac{\alpha_0}{(1+\alpha_0-s)(1-s)} - \frac{\beta_0}{(\beta_0+s)s},$$

so $V''(x) > 0$ and $g$ is strictly convex. Consequently $g(x) > 0$ if $x > R$, which is what we wanted to prove.

Note the asymmetry in (4.23) and (4.24). Just as for the arctic circle large inward fluctuations of the temperate zone have much smaller probability than large outward fluctuations.

We will now compute (a part of) the arctic ellipse. For simplicity we restrict to the case $a = b$. The general case can be handled similarly but the computations are somewhat more involved. One approach is to compute the equilibrium measure, and hence its suport which gives the arctic ellipse, by solving the variational problem (4.22) as was done for the Krawtchouk polynomials in [14]. Below we will instead use the approach of [50] which is based on the recursion formula for the Hahn polynomials.

**Lemma 4.2.** *Let $R(t, \alpha_0)$ be the right endpoint of the support of $u_{\text{eq}}^{(t,\alpha_0,\alpha_0)}$ as defined above. Then,*

$$R(t,\alpha_0) = \frac{1}{t}\sup_{0<s<t}\Big(\frac{1}{2} + \frac{1}{2(s+\alpha_0)}\sqrt{s(1-s)(s+2\alpha_0)(s+2\alpha_0+1)}\Big). \tag{4.26}$$

*Proof.* Let $q_n = q_{n,N}^{(\alpha,\beta)}(x)$ denote the normalized Hahn polynomials which are orthonormal on $\{0, \dots, N\}$ with respect to the weight (4.21),

$$\begin{aligned} q_{n,N}^{(\alpha,\beta)}(x) &= \frac{(-1)^n}{d_{n,N}}\,{}_3F_2(-n,-x,n+\alpha+\beta+1;\beta+1,-N;1) \\ &= \frac{(-1)^n}{d_{n,N}}\sum_{k=0}^{n}\frac{(-n)_k(-x)_k(n+\alpha+\beta+1)_k}{(\beta+1)_k(-N)_k k!}, \end{aligned} \tag{4.27}$$

where $d_{n,N} > 0$ and

$$d_{n,N}^2 = \frac{(\alpha+\beta+1)(\alpha+1)_n(N+\alpha+\beta+2)_n}{\binom{N}{n}(2n+\alpha+\beta+1)(\beta+1)_n(\alpha+\beta+1)_n}.$$

The leading coefficients are

$$\kappa_{n,N} = \frac{(-1)^n}{d_{n,N}}\frac{(n+\alpha+\beta+1)_n}{(\alpha+1)_n(-N)_n}. \tag{4.28}$$

The polynomials $q_n$ satisfy the recurrence relation

$$xq_n = a_{n,N}q_{n-1} + b_{n,N}q_n + a_{n+1,N}q_{n+1},$$

where

$$a_{n,N} = \frac{n(n+\alpha)(n+\alpha+\beta+N+1)}{(2n+\alpha+\beta)(2n+\alpha+\beta+1)}\sqrt{\frac{(N-n+1)(2n+\alpha+\beta+1)(\beta+n)(\alpha+\beta+n)}{(\alpha+n)(n+N+\alpha+\beta+1)n(2n+\alpha+\beta+1)}}$$

and

$$b_{n,N} = \frac{(n+\alpha+\beta+1)(n+\beta+1)(N-n)}{N(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)} + \frac{n(n+\alpha)(n+\alpha+\beta+N+1)}{N(2n+\alpha+\beta)(2n+\alpha+\beta+1)}.$$

Consider the rescaled equilibrium measure, $u_{\text{eq}}^*(s) = \frac{1}{t}u_{\text{eq}}(s/t)$, $0 \le s \le 1$. The paper [50] tells us how to compute the support of $u_{\text{eq}}^*$ (and also the measure itself)

using the asymptotics of the recursion coefficients. We restrict to the case $\alpha_0 = \beta_0$. Then, $n/N \to t \in (0,1)$,

$$\lim_{N\to\infty} a_{n,N} = a(t) = \frac{1}{4(t+\alpha_0)}\sqrt{t(1-t)(t+2\alpha_0)(t+2\alpha_0+1)}$$
$$\lim_{N\to\infty} b_{n,N} = b(t) = \frac{1}{2}.$$

According to [50], p. 171, the right endpoint of the support of $u^*_{\text{eq}}$, i.e. $tR(\alpha_0,t)$ is

$$tR(\alpha_0,t) = \sup_{0<s<t}(b(s)+2a(s)).$$

□

We can now formulate the result we obtain for the arctic ellipse.

**Theorem 4.3.** *Consider the abc-hexagon, assume that $a = b$ and rescale the size of the hexagon by a factor $1/c$. We let the size of the hexagon grow in such a way that $a/c \to \lambda > 0$ as $c \to \infty$. Pick the ON-coordinate system in which the limiting hexagon has corners at $\pm(-\frac{\sqrt{3}}{2}\lambda, \frac{1}{2})$, $\pm(0, \frac{1}{2}(1+\lambda))$ and $\pm(\frac{\sqrt{3}}{2}\lambda, \frac{1}{2})$. Let $X$ be the intersection point of the line $x = \tau$, $-\frac{\sqrt{3}}{2} < \tau < -\frac{\sqrt{3}}{2}\frac{\lambda^2}{\lambda+1}$, with the inner part of the boundary of the rescaled polar zone at $P_6$. Then,*

$$X \to \sqrt{2\lambda+1}\sqrt{1/4-\tau^2/3\lambda^2} \tag{4.29}$$

*in probability as $c \to \infty$. For any $\epsilon > 0$ there are constants $I(\epsilon) > 0$ and $J(\epsilon) > 0$ such that*

$$\frac{1}{c}\log P[X \geq \sqrt{2\lambda+1}\sqrt{1/4-\tau^2/3\lambda^2}+\epsilon] \to -I(\epsilon) \tag{4.30}$$

*and*

$$\frac{1}{c^2}\log P[X \leq \sqrt{2\lambda+1}\sqrt{1/4-\tau^2/3\lambda^2}-\epsilon] \to -J(\epsilon) \tag{4.31}$$

*as $c \to \infty$.*

*Proof.* We are in the case when $0 \leq m \leq a = b$, so $n = L_m = m$, $N = \gamma_m = m + c - 1$, $\alpha = a - m = \beta = b - m$. Assume that $m/c \to \mu > 0$ as $c \to \infty$. Then $t = \mu(1+\mu)^{-1}$ and $\alpha_0 = (\lambda-\mu)(1+\mu)^{-1} = (1-t)\lambda - t$. Consider the points $B_m = (m, \alpha_m + 2Z_m)$ which describe the inner boundary of the polar zone; $\alpha_m = -m$. We see that

$$Z_m/L_m \to R(t,\alpha_0) \tag{4.32}$$

in probability as $c \to \infty$ Here we use the large deviation formulas (4.23) and (4.24) together with 4.1. To get an ON-system (with the sides of the rhombi = 1 we have to rescale the coordinates to $\tilde{B}_m = (\frac{\sqrt{3}}{2}m, -\frac{m}{2}+Z_m)$. Thus,

$$\frac{1}{c}\tilde{B}_m = (\frac{\sqrt{3}}{2}\frac{m}{c}, -\frac{m}{2c}+\frac{L_m}{c}\frac{Z_m}{L_m}) \to (\frac{\sqrt{3}}{2}\mu, -\frac{\mu}{2}+\mu R)$$

We also have to translate the coordinate system so that we get the origin at the center of the hexagon. We then get the coordinates

$$(\frac{\sqrt{3}}{2}(\mu-\lambda), -\frac{\mu+1}{2}+\mu R) \tag{4.33}$$

for the limit of the point $X$ on the arctic ellipse. Now,

$$R(\frac{\mu}{\mu+1},\frac{\lambda-\mu}{\mu+1})=\frac{\mu+1}{2\mu}+\frac{1}{2\lambda\mu}\sqrt{(2\lambda+1)\mu(2\lambda-\mu)} \tag{4.34}$$

for $0\le\mu\le\lambda(\lambda+1)^{-1}$. To see this we use lemma 4.2 and set

$$g(s)=\frac{1}{2}+\frac{1}{2(s+\alpha_0)}\sqrt{s(1-s)(s+2\alpha_0)(s+2\alpha_0+1)}$$

with $\alpha_0=(1-t)\lambda-t$. We must have $g(s)\le 1$ because the support is restricted to $[0,1/t]$. A computation shows that $g(s)$ is strictly increasing in $[0,s_0]$ and strictly decreasing in $[s_0,1]$ for some $s_0\in(0,1)$ For $0\le t\le s_0$,

$$\sup_{0<s<t} g(s)=g(t)=\frac{1}{2}+\frac{2\lambda+1}{2\lambda}\sqrt{t(\frac{2\lambda}{2\lambda+1}-t)}, \tag{4.35}$$

where we have inserted $\alpha_0=(1-t)\lambda-t$. Suppose $s_0<\lambda(2\lambda+1)^{-1}$. Then $g(s_0)>g(\lambda(2\lambda+1)^{-1})=1$, which is impossible. Thus (4.35) holds for $0\le t\le\lambda(2\lambda+1)^{-1}$ and (4.34) follows from (4.26) since $t=\mu(\mu+1)^{-1}$. Set $\tau=\frac{\sqrt{3}}{2}(\mu-\lambda)\in(-\frac{\sqrt{3}}{2},-\frac{\sqrt{3}}{2}\frac{\lambda^2}{\lambda+1})$. Then the lmiting point (4.33) becomes $(\tau,\sqrt{2\lambda+1}\sqrt{1/4-\tau^2/3\lambda^2})$. The large deviation formulas (4.30) and (4.31) follow from (4.23) and (4.24). □

The 1-dimensional marginal probability in the Hahn ensemble (4.3) is, [52],

$$u_{N,n}^{(\alpha,\beta)}(t)=\frac{1}{n}\sum_{k=0}^{n-1}q_{k,N}^{(\alpha,\beta)}(t)^2W_N^{(\alpha,\beta)}(t), \tag{4.36}$$

for $t\in\{0,\dots,N\}$, where $q_{k,N}^{(\alpha,\beta)}$ are the normalized Hahn polynomials (4.27). Hence, the probability of finding a hole at position $t$, i.e. the 1-point correlation function, is $nu_{N,n}^{(\alpha,\beta)}(t)$, and consequently the number of rhombus tilings of the abc-hexagon with a vertical rhombus at position $t$ on the line $x=m$ is

$$N(a,b,c)\sum_{k=0}^{L_m-1}q_{k,N}^{(\alpha,\beta)}(t)^2W_N^{(a_m,b_m)}(t), \tag{4.37}$$

by theorem 4.1. (This can be rewritten using the Christoffel-Darboux formula.) If we use the explicit formula (4.27) for the Hahn polynomials, we obtain an explicit formula for the quantity (4.37). This quantity has been investigated in [18], [24], [25]. We will not attempt to rewrite (4.37) in oeder to compare it with existing formulas. We can also consider the number of tilings with vertical rhombi at specified positions $t_1,\dots,t_r$ on the line $x=m$. This is given by $N(a,b,c)$ times a determinantal correlation function like (2.10) but where we now have instead the Hahn kernel given by the formula (2.11) with the Hahn polynomials and the Hahn weight instead. It follows from the general theory in [37] that the 1-dimensional marginal probability converges weakly to the equilibrium measure. This should also hold pointwise, i.e.

$$\lim_{N\to\infty}u_{N,n}^{(\alpha,\beta)}([n\tau])=u_{\mathrm{eq}}^{(t,\alpha_0,\beta_0)}(\tau), \tag{4.38}$$

if $\frac{1}{N}(\alpha,\beta)\to(\alpha_0,\beta_0)$ and $n/N\to t$ as $N\to\infty$, but we do not have good enough control over the asymptotics of the Hahn polynomials to prove it. The corresponding result for the Krawtchouk ensemble follows by the same methods as was used to prove lemma 2.8.

As mentioned in the beginning of this section a rhombus tiling of an abc-hexagon can also be interpreted as a boxed planar partition where the sides of the box are $a$, $b$ and $c$. We want to relate the height of the planar partition surface above a certain point to our particle configurations. Let $(x, y, z)$ be the coordinates in the planar partition coordinate system (the $x$-axis through $P_1$, the $y$-axis through $P_3$ and the $z$-axis through $P_5$ after projection). Consider the line $x = m$. We start from the point $(r_m, s_m, 0)$, where

$$r_m = \begin{cases} a & \text{if } 0 \le m \le b \\ a + b - m & \text{if } b \le m \le a + b \end{cases}$$

and

$$s_m = \begin{cases} m & \text{if } 0 \le m \le b \\ b & \text{if } b \le m \le a + b \end{cases}.$$

As we go along the line a particle means that the $z$-coordinate is increased by 1, whereas the $x$- and $y$-coordinates are fixed. A hole means that the $z$-coordinate is fixed, but the $x$- and $y$-coordinates are reduced by 1. Let $X_m(k)$ denote the position of hole number $k$. This hole corresponds to the position $(r_m - k, s_m - k)$ in the $xy$-plane and the surface height above this point is equal to the number of of particles in $\{0, \dots, X_m(k)\}$, which equals $X_m(k) - k + 1$. Thus, the planar partition height function is given by

$$H(r_m - k, s_m - k) = X_m(k) - k + 1. \tag{4.39}$$

Let $Y_m(n)$ denote the number of holes in $[0, n]$ in the particle system on $x = m$. Then

$$P[X_m(k) \le n] = 1 - P[Y_m(n) < k]. \tag{4.40}$$

Assume that $m/c \to \mu > 0$, $L_m/\gamma_m \to t > 0$, $n/L_m \to \tau$ and $\gamma^{-1}(|a-m|, |b-m|) \to (\alpha_0, \beta_0)$, as $c \to \infty$. Then theorem 4.1 and the general theory of discrete Coulomb gases in [37] shows that

$$\frac{1}{L_m} E[Y_m(n)] \to \int_0^\tau u_{\text{eq}}^{(t,\alpha_0,\beta_0)}(s)ds \tag{4.41}$$

as $c \to \infty$. Also, it is possible to prove large deviation results, analogous to those in [4] for the semi-circle law, in this case too. Using this and (4.39), (4.40) and (4.41) it is possible to compue the asymptotic shape of the planar partition surface (law of large numbers) as well as large deviation results. See [12] for this type of results proved by other methods. The asymptotic shape can be computed if we know the equilibrium measure.

From (4.39) and (4.40) it is clear that the surface fluctuations are directly related to the fluctuations of the number of paticles in an interval in the Hahn ensemble. Hence, in analogy with the Krawtchouk case, the surface fluctuations should be Gaussian with variance proportional to $\log c$ as $c \to \infty$. This could be proved provided we had the same control of the Hahn kernel as we have of the Krawtchouk kernel in lemma 2.8, but this remains to be done.

We will end this section by showing one way of finding GUE in a random rhombus tiling of an abc-hexagon in the limit as the size of the hexagon goes to infinity. Consider the $m$ holes in the $m$:th column, $1 \le m \le b = a$. we want to compute the probability distribution of these $m$ holes, with $m$ fixed, as the hexagon grows.

Let $\xi_1,\dots,\xi_m$ be the positions of the holes. By theorem 4.1 the probability of this configuration is

$$\frac{1}{Z^{(a-m,a-m)}_{\gamma_m,m}}\Delta_m(\xi)^2\prod_{j=1}^m\frac{(\gamma_m+a-m-\xi_j)!(a-m+\xi_j)!}{\xi_j!(\gamma_m-\xi_j)!}, \tag{4.42}$$

where $\gamma_m=m+c-1$. Denote the corresponding expectation by $E^{(a-m,a-m)}_{\gamma_m,m}[\cdot]$.

**Proposition 4.4.** *Let $c\to\infty$, $a/c\to\lambda>0$ and keep $m\ge 1$ fixed. Let $f:\mathbb{R}^m\to\mathbb{C}$ be a continuous, bounded, symmetric function. Then,*

$$\begin{aligned}&\lim_{c\to\infty}E^{(a-m,a-m)}_{\gamma_m,m}[f(\frac{\xi_1-\gamma_m/2}{\sqrt{c}},\dots,\frac{\xi_m-\gamma_m/2}{\sqrt{c}})]\\ &=\frac{1}{Z_m(\lambda)}\int_{\mathbb{R}^m}\Delta_m(x)^2\prod_{j=1}^m e^{-\frac{2\lambda}{\lambda+1}x_j^2}f(x_1,\dots,x_m)d^mx,\end{aligned} \tag{4.43}$$

*where $Z_m(\lambda)$ is a normalization constant such that the right hand side is 1 when $f=1$.*

Thus, in the limit, the positions of the holes (vertical rhombi) on the $m$:th vertical column are described by $m\times m$ GUE.

*Proof.* The proof is straightforward. By (4.42) the expectation in the left hand side of (4.43) equals

$$\begin{aligned}\frac{1}{Z^{(a-m,a-m)}_{\gamma_m,m}}&\sum_{\xi\in\{0,\dots,\gamma_m\}^m}\Delta_m(\xi)^2 f(\frac{\xi_1-\gamma/2}{\sqrt{c}},\dots,\frac{\xi_m-\gamma/2}{\sqrt{c}})\\ &\qquad\times\prod_{j=1}^m\frac{(\gamma+a-m-\xi_j)!(a-m+\xi_j)!}{\xi_j!(\gamma_m-\xi_j)!}.\end{aligned}$$

In this expression we write $\xi_j=n_j+\gamma/2$ and use Stirling's formula to approximate the factorials. This leads to an expression which is a Riemann sum. The normalization constant is the same Riemann sum but with $f=1$. After cancelling common factors we see that the remaining quotient of Riemann sums converges to the right hand side of (4.43). □

## 5. A dimer model on a cylindrical brick lattice

In this final section we will consider a dimer model on a hexagonal graph on a finite cylinder. The graph is sometimes referred to as the brick lattice, [69]. A dimer covering of this lattice can also be thought of as a certain cylindrical rhombus tiling, see [45]. The dimer covering has an equivalent description in terms of non-intersecting random walk paths, [16]. In contrast to the previous non-intersecting path models we do not have a fixed number of paths. We will again be interested in the point process we obtain by looking at where the random walks are at a fixed time, and the analysis is based on the methods of [67].

The graph $G_{M,N}$ which we will consider is defined as follows. The vertices are $v_{j,k}=(-1/2+j,k)$, $0\le j\le 2M-1$, $0\le k\le 2N$ and we obtain a graph on a cylinder by identifying $v_{j,k}$ and $v_{j+2m,k}$ for all $j,k$. We have vertical edges between $v_{j,k}$ and $v_{j,k+1}$, and horizontal edges between $v_{2j,2k}$ and $v_{2j+1,2k}$ and between $v_{2j+1,2k+1}$ and $v_{2j+2,2k+1}$. A dimer covering of $G_{M,N}$ can equivalently be described by non-intersecting random walk paths, [16],[53]. In the same coordinate system

as was used to define $G_{M,N}$ we have simple random walk paths $S(t)$ with steps $\pm 1$, with initial position $S(0) \in \{2k\,;\, 0 \le k \le N\}$ and which are required to satisfy $0 \le S(t) \le 2N$. Since we have a graph on a cylinder we and the paths to live on the cylinder also so we require that $S(t+2M) = S(t)$. We have $L$ non-intersecting paths, where $0 \le L \le N$. (Actually the condition $0 \le S(t) \le 2N$ implies that $L \le N-1$.) Let $\mathcal{P}_{M,N}$ be th set of all such families of non-intersecting paths. We now describe the 1-1-correspondence between $\mathcal{P}_{M,N}$ and the set of all dimer configurations on $G_{M,N}$. A dimer covers a vertical edge if and only if we have a random walk step, in one of the paths, which intersects the vertical edge. A dimer covers a horizontal edge if and only if *no* random walk path hits this edge. If the horizontal edge $v_{2j,2k}v_{2j+1,2k}$ is *not* covered by a dimer, then one of $e_1 = v_{2j+1,2k}v_{2j+1,2k+1}$ or $e_2 = v_{2j+1,2k}v_{2j+1,2k-1}$ must be covered. A random walk path must pass through $(2j, 2k)$ and in the next step intersect either $e_1$ or $e_2$. It will go to either $(2j+1, 2k+1)$ or $(2j+1, 2k-1)$ which means that we obtain a new horizontal edge that is not covered by a dimer and we can repeat the argument. A random walk path connot inresect both $e_1$ and $e_2$ because then they would have to meet at $(2j, 2k)$ which would contradict the non-intersection condition. A similar argument applies at all locations in the graph. Hence, a dimer configuration gives rise to non-intersecting paths, and conversely if we have non-intersecting paths we can find the dimer covering corresponding to them.

Consider a dimer covering of $G_{M,N}$. The total number of vertical dimers is $2ML$, where $L$ is the number of non-intersecting paths in the path description. The total number of dimers is $M(2N+1)$ and hence the numbeer of horizontal dimers is $M(2N+1-2L)$. We will now define a probability measure on the set of dimer configurations by letting vertical dimers have weight $w$ and horizontal dimers weight $z$. Let $g_{M,N}(m,n)$ denote the number of configurations with $m$ horizontal and $m$ vertical dimers; each of these have the same probability. The *partition function* is given by

$$Z = Z_{M,N}(z,w) = \sum_{m,n \ge 0} g_{M,N}(m,n) z^m w^n. \tag{5.1}$$

If $G_L$ is the number of non-intersecting path configurations with exactly $L$ paths, then we must also have

$$Z = \sum_{L=0}^{N} G_L z^{M(2N+1-2L)} w^{2ML} = z^{M(2N+1)} \sum_{L=0}^{N} G_L \left(\frac{w}{z}\right)^{2ML}. \tag{5.2}$$

Recall that the possible initial (=final) positions for the non-intersecting paths are $\{2k\,;\, 0 \le k \le N\}$. Let $G_L(x)$ denote the number of configurations with $L$ non-intersecting paths whose initial conditions are $2x_1 < \cdots < 2x_L$, where $x_1, \dots, x_L \in [N] \doteq \{0, \dots, N\}$ are given. Then,

$$G_L = \sum_{0 \le x_1 < \cdots < x_L \le N} G_L(x). \tag{5.3}$$

Hence, the probability that the initial positions $x$ *given* that we have exactly $L$ non-intersecting paths is $G_L(x)/G_L$, and we define a probability on $[N]^L$ by

$$u_L(x) = \frac{1}{L! G_L} G_L(x), \tag{5.4}$$

where the right hand side is extended to $[N]^L$ by requiring it to be a symmetric function. The *l-particle correlation function*, *given* that the total number of

particles is $L$, is defined by

$$R_{\ell,L}(x_1,\dots,x_\ell)=\frac{L!}{(L-\ell)!}\sum_{x_{\ell+1},\dots,x_L\in[N]}u_L(x_1,\dots,x_L). \tag{5.5}$$

The probability of having exactly $L$ particles is, by (5.2)

$$\frac{G_L}{Z}(w/z)^{2ML}z^{M(2N+1)}$$

and hence the *$\ell$-particle correlation function*, with no restriction on the total number of particles, is

$$R_\ell(x_1,\dots,x_\ell)=\frac{z^{M(2N+1)}}{Z}\sum_{L=\ell}^{N}R_{\ell,L}(x_1,\dots,x_\ell)(w/z)^{2ML}G_L. \tag{5.6}$$

Set

$$\phi(s,t)=\sqrt{\frac{2c_{2s}c_{2t}}{N}}\cos\frac{\pi st}{N}, \tag{5.7}$$

$0\le s,t\le N$, where $c_m=1/2$ if $m=0$ or $m=2N$ and $c_m=1$ if $1\le m\le 2N-1$.

**Proposition 5.1.** *Set*

$$K(x,y)=\sum_{j=0}^{N}\phi(x,j)\phi(y,j)\frac{(2w/z)^{2M}w_j}{1+(2w/z)^{2M}w_j}, \tag{5.8}$$

*where* $w_j=(\cos\frac{\pi j}{2N})^{2M}$. *Then,*

$$R_\ell(x_1,\dots,x_\ell)=\det(K(x_i,x_j))_{i,j=1}^{\ell}. \tag{5.9}$$

*Also,*

$$Z_{M,N}(z,w)=z^{M(2N+1)}\prod_{k=0}^{N}(1+(\frac{2w}{z}\cos\frac{\pi k}{2N})^{2M}). \tag{5.10}$$

*Proof.* We consider a simple random walk on $\{-1,0,\dots,2N+1\}$, where $-1$ and $2N+1$ are absorbing barriers. The transition matrix is given by $P^*_{n,n}=0$ if $0\le n\le 2N$, $P^*_{n,n}=1$ if $n=-1$ or $n=2N+1$, $P^*_{n,n-1}=1/2$ if $1\le n\le 2N+1$ $P^*_{n,n+1}=1/2$ if $0\le n\le 2N-1$ and $P^*_{m,n}=0$ otherwise. Since we are only interested in random walk paths that are not absorbed we can concentrate on the submatrix $P=(P^*_{m,n})_{0\le m,n\le 2N}$. We are interested in the probability of going from $2x$ to $2y$ in $2M$ steps, which is given by $(P^{2M})_{2x,2y}$. A computation, see [41], shows that

$$\sum_{j=0}^{N}(P^{2M})_{2x,2j}\phi(j,k)=w_k\phi(x,k)$$

and because of the orthogonality

$$\sum_{j=0}^{N}\phi(s,j)\phi(j,t)=\delta_{st},$$

we obtain

$$(P^{2M})_{2x,2y}=\sum_{j=0}^{N}w_j\phi(x,j)\phi(j,y), \tag{5.11}$$

with $w_j$ as above.

Now, it follows from the Karlin/McGregor theorem that

$$G_L(x) = 2^{2ML} \det((P^{2M})_{2x_i,2x_j})_{i,j=1}^L, \tag{5.12}$$

where the factor $2^{2ML}$ comes from the fact that $G_L(x)$ counts the *number* of configurations. Note that the right hand side of (5.12) is a symmetric function of $x_1, \dots, x_L$. Let $f$ be a given function on $\mathbb{N}$ and set

$$\Phi_L[f] = \sum_{x \in [N]^L} \det((P^{2M})_{2x_i,2x_j})_{i,j=1}^L \prod_{j=1}^L f(x_j), \tag{5.13}$$

and write $[\lambda^\ell]F(\lambda)$ for the coefficient of $\lambda^\ell$ in the power series $F(\lambda)$. It follows from (5.4), (5.5), (5.12) and (5.13) that

$$\sum_{x \in [N]^L} R_{\ell,L}(x_1, \dots, x_\ell) \prod_{j=1}^L f(x_j) = \frac{\ell! 2^{2ML}}{L! G_L} [\lambda^\ell] \Phi_L[1 + \lambda f]. \tag{5.14}$$

Also, we set

$$\Phi[f] = \sum_{L=0}^N \frac{(2w/z)^{2ML}}{L!} \Phi_L[f] \tag{5.15}$$

and note that by (5.2), (5.3), (5.12) and (5.13)

$$\Phi[1] = \frac{Z}{z^{M(2N+1)}}. \tag{5.16}$$

Now, by (5.6), (5.14), (5.15) and (5.16),

$$\sum_{x \in [N]^\ell} R_\ell(x_1, \dots, x_\ell) \prod_{j=1}^\ell f(x_j) = \ell! [\lambda^\ell] \frac{\Phi[1 + \lambda f]}{\Phi[1]}. \tag{5.17}$$

By (5.11) and a classical identity, see [67],

$$\begin{aligned} \det((P^{2M})_{2x_i,2x_j})_{i,j=1}^L &= \det(\sum_{k=0}^N w_k \phi(x_i, k) \phi(k, x_j))_{i,j=1}^L \\ &= \frac{1}{L!} \sum_{k \in [N]^L} [\det(\phi(x_i, k_j)_{i,j=1}^L]^2 \prod_{j=1}^L w_{k_j}. \end{aligned}$$

Thus, if we use the same identity again in the other direction we obtain

$$\Phi_L[f] = \sum_{k \in [N]^L} \det(\sum_{x=0}^N w_{k_i} \phi(x, k_i) \phi(x, k_j) f(x))_{i,j=1}^L. \tag{5.18}$$

Set

$$\mathcal{K}(s,t) = \sum_{x=0}^N w_s \phi(x,s) \phi(x,t) f(x).$$

Then, by (5.15), (5.18) and a Fredholm expansion

$$\Phi[f] = \det(\delta_{i,j} + (2w/z)^{2M} \mathcal{K}(i,j))_{i,j=0}^N. \tag{5.19}$$

Set $A = ((1 + (2w/z)^{2M} w_i)\delta_{i,j})_{i,j=0}^N$ and $B = ((2w/z)^{2M}\mathcal{K}(i,j))_{i,j=0}^N$. Then, by (5.19),

$$\Phi[1+\lambda f] = \det(A+\lambda B). \tag{5.20}$$

Combining (5.16) and (5.20) we obtain (5.10). Furthermore, (5.17) can be written

(5.21)

$$\sum_{x\in[N]^\ell} R_\ell(x_1,\dots,x_\ell)\prod_{j=1}^{\ell} f(x_j) = \ell![\lambda^\ell]\det(I+\lambda A^{-1}B) = \ell![\lambda^\ell]\det(I+\lambda Kf)$$

with $K$ given by (5.8). The last step is proved as in [67], see also [39]. Expanding the last expression in (5.21) in a Fredholm expansion gives (5.9) since $f$ was arbitrary, and the proposition is proved □

We will now discuss the asymptotics of the partition function and the correlation functions. It is well known how to obtain a formula like (5.10) for the partition function using Kasteleyn's method, [44]. Also the two-point correlation function has been computed in [69] using the methods of [17]. See also [20]. The computation above is different and emphasizes the similarity between certain aspects of the dimer model and random matrix theory. The limiting correlation functions we obtain are given by the discrete or ordinary sine kernel.

The number of vertices is $2M(2N+1)$ and the *free energy* per vertex is defined by

$$f_{M,N}(z,w) = \frac{1}{2M(2N+1)}\log Z_{M,N}(z,w). \tag{5.22}$$

**Proposition 5.2.** *The limiting free energy is*

$$\begin{aligned} f(z,w) &= \lim_{M\to\infty}\lim_{N\to\infty} f_{M,N}(z,w) \\ &= \begin{cases} \frac{1}{2}\log z & \text{if } w/z < 1/2 \\ \frac{1}{2}\log z + \frac{1}{2\pi}\int_0^{\pi\theta_0/2}\log(\frac{2w}{z}\cos s)ds & \text{if } w/z > 1/2, \end{cases} \end{aligned} \tag{5.23}$$

*Proof.* By (5.10) and (5.22) we obtain

$$\begin{aligned} \lim_{N\to\infty} f_{M,N}(z,w) &= \frac{1}{2}\log z + \frac{1}{4M}\lim_{N\to\infty}\frac{1}{N+1}\sum_{k=0}^{N}\log(1+(\frac{2w}{z}\cos\frac{\pi k}{2N})^{2M}) \\ &= \frac{1}{2}\log z + \frac{1}{2\pi M} + \int_0^{\pi/2}\log(1+(\frac{2w}{z}\cos s)^{2M})ds \end{aligned}$$

If $w/z < 1/2$, then the integrand goes to 0 uniformly and we obtain the first part of (5.23). If $w/z > 1/2$, then the integrand goes to 0 unless $0 \le s \le \arccos(z/2w) = \pi\theta_0/2$ and we obtain our result by using the inequalities

$$(\frac{2w}{z}\cos s)^{2M} \le 1 + (\frac{2w}{z}\cos s)^{2M} \le 2(\frac{2w}{z}\cos s)^{2M} \tag{5.24}$$

for $0 \le s \le \theta_0$. □

If we only had horizontal dimers ($L = 0$), then $Z = z^{M(2N+1)}$ and hence the free energy per vertex is $\frac{1}{2}\log z$. We can thus interpret the phase transition in (5.23) as saying that for $w/z < 1/2$ the system is completely frozen, whereas for $w/z > 1/2$ we have both horizontal and vertical dimers. This type of phase transition is called a *K-type transition* in [53]. We will study the limiting correlation functions in the

non-frozen phase $w/z > 1/2$ and in a scaling limit where we approach the critical point from above. In this scaling limit we will obtain the sine kernel determinantal point process of random matrix theory. By proposition 5.1, formula (5.9) it suffices to investigate the asymptotics of the kernel (5.8).

**Proposition 5.3.** *Assume that $w/z > 1/2$. Then*

$$\lim_{M\to\infty}\lim_{N\to\infty} K(\frac{N}{2}+t,\frac{N}{2}+s) = \frac{\sin\pi(t-s)\theta_0}{\pi(t-s)}, \tag{5.25}$$

*for any fixed $t, s \in \mathbb{Z}$. Here $\theta_0 = \frac{2}{\pi}\arccos\frac{z}{2w}$ as before. Let $\epsilon_N$, $N \geq 1$, be a given sequence such that $\epsilon \to 0$ and $N\sqrt{\epsilon_N} \to \infty$ as $N \to \infty$. Assume furthermore that $M = M(N) \to \infty$ and $M\epsilon_N \to \infty$ as $N \to \infty$. If $2w/z = 1 + \epsilon_N$, then*

$$\lim_{N\to\infty} \frac{\pi}{2\sqrt{2\epsilon_N}} K(\frac{N}{2}+[\frac{\pi\xi}{2\sqrt{2\epsilon_N}}],\frac{N}{2}+[\frac{\pi\eta}{2\sqrt{2\epsilon_N}}]) = \frac{\sin\pi(\xi-\eta)}{\pi(\xi-\eta)}, \tag{5.26}$$

*for any fixed $\xi, \eta \in \mathbb{R}$.*

Note that $\theta_0$ gives the local density of the intersection of the random walk paths with a vertical axis and that $\theta \to 0$ as $w/z \to 1/2+$.

*Proof.* Assume that $N = 2n$ for simplicity. Then, by the definition of $K(x, y)$,

$$\begin{aligned}&\lim_{N\to\infty} K(n+t,n+s)\\ &\lim_{n\to\infty}\frac{1}{n}\sum_{j=0}^{2n}\cos(\frac{\pi j}{2}+\frac{\pi tj}{2n})\cos(\frac{\pi j}{2}+\frac{\pi sj}{2n})\frac{(\frac{2w}{z}\cos\frac{\pi j}{4n})^{2M}}{1+(\frac{2w}{z}\cos\frac{\pi j}{4n})^{2M}}.\end{aligned}$$

Split the last sum into two depending on whether $j$ is even or odd and compute the limits of the Riemann sums obtained. This gives

$$\int_0^1 [\sin\pi tu\sin\pi su+\cos\pi tu\cos\pi su]\frac{(\frac{2w}{z}\cos\frac{\pi u}{2})^{2M}}{1+(\frac{2w}{z}\cos\frac{\pi u}{2})^{2M}}du.$$

With $\theta_0$ as defined above we see that the limit of this expression as $M \to \infty$ is

$$\int_0^{\theta_0}\cos\pi(t-s)udu = \frac{\sin\pi(t-s)\theta_0}{\pi(t-s)},$$

and we have proved (5.25). Write $\gamma_N = 1 + \epsilon_N = 2w/z$. We have that

$$\begin{aligned}K(\frac{N}{2}+t,\frac{N}{2}+s) \approx \frac{2}{N}\sum_{j=1}^{N/2}\Bigg[&\sin(\frac{\pi t(2j-1)}{N})\sin(\frac{\pi s(2j-1)}{N})\left(\frac{\gamma_N\cos\frac{\pi s(2j-1)}{2N}}{1+\gamma_N\cos\frac{\pi s(2j-1)}{2N}}\right)^{2M}\\ &+\cos\frac{2\pi tj}{N}\cos\frac{2\pi sj}{N}\left(\frac{\gamma_N\cos\frac{\pi j}{N}}{1+\gamma_N\cos\frac{\pi j}{N}}\right)^{2M}\Bigg],\end{aligned} \tag{5.27}$$

where the error is negligible for large $N$. Note that $(1 + \epsilon_N)\cos\frac{\pi j}{N} \geq 1$ if (approximately) $j \leq N\sqrt{2\epsilon_N}/\pi$. Hence the summation in (5.27) can be restricted to $1 \leq j \leq N\sqrt{2\epsilon_N}/\pi$ and in the limit we are considering the right hand side of (5.27) becomes

$$\frac{\pi}{\sqrt{2}}\int_0^{\sqrt{2}/\pi}\cos(\frac{\pi^2}{\sqrt{2}}(\xi-\eta)u)du = \frac{\sin\pi(\xi-\eta)}{\pi(\xi-\eta)},$$

□

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